Different approaches to calculate the $K^{\pm} \to \pi^{\pm}\pi^{0}e^{+}e^{-}$ decay width

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Introduction

The rare $K^{\pm} \to \pi^{\pm}\pi^{0}e^{+}e^{-}$ decay is currently under analysis by the NA62 Collaboration at SPS,CERN.
This motivated us to performed two theoretical approaches to calculate the differential decay width of the decay $K^{+} \to \pi^{+}\pi^{0}e^{+}e^{-}$:
1. In the kaon rest frame, where we use the five variables introduced by Cabibbo-Maksimovicz(1965) for $K^{+} \to \pi^{+}\pi^{0}e^{+}\nu$ decay.
2. In the center-of-mass system of the lepton pair, which essentially simplifies the computations.
A comparison between the two approaches has been performed and the dependencies of the differential decay rate as a function of the virtual photon and dipion system masses were investigated.

Radiative kaon decay $K^{\pm} \to \pi^{\pm}\pi^{0}\gamma$

For many years radiative decay has been considered as a good tool for studying the low energy structure of QCD.The amplitude of this process consists of two parts:

1. Long distance contribution called inner Bremsstrahlung (IB) associated with the $K^{\pm} \to \pi^{\pm}\pi^{0}$ decay according to Low's(1958) theorem.

2.Direct emission (DE) part (decomposed into electric and magnetic parts) can be calculated in the framework of the Chiral Perturbation Theory (ChPT).

As the DE piece is almost two order of magnitude smaller than the Bremsstrahlung contribution its determination is " state of the art" problem for experiment and theory.

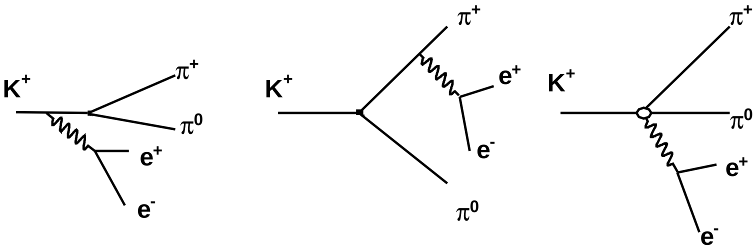

Figure : *The first two diagrams represent the inner Bremsstrahlung contribution. The third diagram corresponds to the direct emission.*

The amplitude and decay width

The invariant amplitude of the decay can be parameterized as a product of leptonic and hadronic currents :

$$A = \frac{e}{q^2} j^\mu(k_1, k_2) J_\mu(p_1, p_2, q)$$

where p_1, p_2 are the 4-momenta of charged and neutral pions, k_1, k_2 – the leptons 4-momenta and $q = k_1 + k_2$ is the momentum of the virtual photon. The leptonic current is:

$$j^\mu(k_1, k_2) = \bar{u}(k_2)\gamma^\mu v(k_1)$$

whereas the hadronic current is represented in terms of two electric form factors $F_{1,2}$ and the magnetic one F_3:

$$J_\mu(p_1, p_2, q) = F_1 p_{1\mu} + F_2 p_{2\mu} + F_3 \epsilon^{\mu\nu\alpha\beta} p_{1\nu} p_{2\mu} q_\beta$$

The decay width is given by the standard expression:

$$d\Gamma = \frac{1}{2M_k}|A|^2 d\Phi$$

The invariant phase space for the four-body decay is usually defined as:

$$d\Phi = (2\pi)^4\delta(P_k - p_1 - p_2 - k_1 - k_2)\frac{d^3p_1}{2(2\pi)^3E_1}\frac{d^3p_2}{2(2\pi)^3E_2}\frac{d^3k_1}{2(2\pi)^3\varepsilon_1}\frac{d^3k_2}{2(2\pi)^3\varepsilon_2}$$

The square of the leptonic current summed over spins:

$$t^{\mu\nu} = \sum_{spins} j^\mu j^\nu = 2(q_\mu q_\nu - k_\mu k_\nu - q^2 g^{\mu\nu})$$

where $k = k_1 - k_2$ is the difference of leptons momenta, m_e, m, m_0 are electron, charged and neutral pion masses, correspondingly.

Introducing the relevant variables for the dipion as $P = p_1 + p_2$ and $Q = p_1 - p_2$ one obtains:

$$
\begin{aligned}
|A|^2 &= \frac{e^2}{q^4}\left(|A_E|^2 + |A_M|^2 + A_{EM}\right); \\
|A_E|^2 &= (-4m^2q^2 + (qP + qQ)^2 - (kP + kQ)^2)|F_1|^2 \\
&+ (-4m_0^2q^2 + (qP - qQ)^2 - (kP - kQ)^2)|F_2|^2 \\
&+ (F_1F_2^* + F_1^*F_2)(-q^2(P^2 - Q^2) + (qP)^2 + (kQ)^2 - (qQ)^2 - (kP)^2) \\
|A_M|^2 &= |F_3|^2\{m_e^2[(16m^2m_0^2 - (P^2 - Q^2)^2)q^2 - 4m^2((qP)^2 + (qQ)^2)) \\
&- 4m_0^2((qP)^2 - (qQ)^2)) + 2(P^2 - Q^2)((qP)^2 - (qQ)^2)] \\
&+ \frac{1}{4}(kP + kQ)^2((qp - qQ)^2 - 4m_0^2q^2)) + \frac{1}{4}(kP - kQ)^2((qP + qQ)^2 \\
&+ 2((kP)^2 - (kQ)^2)(q^2P^2 - q^2Q^2 - (qP)^2 + (qQ)^2)\}; \\
A_{EM} &= ((kP + kQ)(F_1^*F_3 + F_1F_3^*) + (kP - kQ)(F_2^*F_3 + F_2F_3^*)) \\
&\times \epsilon^{\mu\nu\rho\sigma}k_\mu q_\nu P_\rho Q_\sigma
\end{aligned}
$$

Hadronic form factors

The electric form factors can be decomposed into Bremsstrahlung and direct emission pieces: $F_i = F_i^B + F_i^{DE}$ while the magnetic form factor consists of direct emission only $F_3 = F_3^{DE}$.
Taking into consideration Low's theorem, the Bremsstrahlung part can be written in terms of the matrix element for the kaon decay into two pions $M(K^+ \to \pi^+\pi^0)$ and the sum of amplitudes corresponding to radiation of the virtual photon by $K^\pm$–meson or charged pion:

$$M(K^+ \to \pi^+\pi^0\gamma^*)_B = eM(K^+ \to \pi^+\pi^0) \times \left(-\frac{\epsilon_\mu P_k^\mu}{(P_k \cdot q) - \frac{q^2}{2}} + \frac{\epsilon_\mu p_1^\mu}{(p_1 \cdot q) + \frac{q^2}{2}} \right)$$

Comparing this expression with hadronic current, one obtains relations between electric form factors F_1^B, F_2^B and decay amplitude $M(K^+ \to \pi^+\pi^0)$:

$$\begin{aligned} F_1^B &= \frac{2ie(qP - qQ)}{(q^2 + qQ + qP)(q^2 + 2qP)} M(K^+ \to \pi^+\pi^0), \\ F_2^B &= \frac{-2ie}{q^2 + 2qP} M(K^+ \to \pi^+\pi^0) \end{aligned}$$

The matrix element of the $K^+ \to \pi^+\pi^0$ decay and form factors caused by direct emission can be calculated in ChPT (Pichl(2001),Cappiello et al.(2012):

$$
\begin{aligned}
M(K^+ \to \pi^+\pi^0) &= \left(\frac{5}{3}G_{27}f_\pi(m_k^2 - m^2) - f_\pi\delta m^2(G_8 + \frac{3G_{27}}{2})\right) e^{i\delta_0^2} \\
&= |M(K^+ \to \pi^+\pi^0)|e^{i\delta_0^2} \\
F_1^{DE} &= -\frac{ieG_8 e^{\delta_1^1}}{f_\pi}\left((qP - qQ)N_E^0 + \frac{4q^2 N_E^1}{3} + 4q^2 L_9\right), \\
F_2^{DE} &= \frac{ieG_8 e^{\delta_1^1}}{f_\pi}\left((qP + qQ)N_E^0 - \frac{2q^2 N_E^{(2)}}{3}\right), \\
F_3^{DE} &= -\frac{2eG_8 e^{\delta_1^1}}{f_\pi}N_M^0, \\
\delta m^2 &= m^2 - m_0^2
\end{aligned}
$$

Here δ_0^2 and δ_1^1 are strong phases associated with the interactions of pions in the final state.
These equations allows one to calculate the differential decay width of the $K^\pm \to \pi^\pm\pi^0 e^+e^-$

Kaon rest frame

To describe the kaon decays to four particles in the final state, it is enough to use five independent variables as it was shown for the $K^{\pm} \to \pi^+\pi^- e^{\pm}\nu$ (K_{e4}) decay many years ago by Cabibbo and Maksymowicz(1965)

Similarly to the K_{e4} channel, we introduce five independent variables which describe completely decay dipion and dilepton invariant masses, $s_\pi = (p_1 + p_2)^2$ and $s_e = q^2 = (k_1 + k_2)^2$, and three angles: θ_π – the angle of the $\pi^{\pm}$ in the $(\pi^{\pm}\pi^0)$ c.m.s with respect to the dipion flight direction; θ_e – the angle of the e^+ in the (e^+e^-) c.m.s with respect to the dilepton flight direction and φ – the angle between dipion and dilepton planes.

Applying Lorentz transformations, we express the covariant scalar products in above expression in terms of these variables:

$$
\begin{aligned}
qP &= \frac{m_k^2 - s_\pi - s_e}{2}; \\
qQ &= (qP)\cdot\frac{\delta m^2}{s_\pi} + \frac{\beta_\pi \lambda^{\frac{1}{2}}(s_\pi, m^2, m_K^2)}{2} cos\theta_\pi; \\
kP &= \frac{1}{2}\beta_e \lambda^{\frac{1}{2}}(s_\pi, m_K^2, q^2) cos\theta_e
\end{aligned}
$$

$$
\begin{aligned}
kQ &= \beta_e cos\theta_e [\frac{\delta m^2}{s_\pi}\frac{\lambda^{\frac{1}{2}}(s_\pi, m_K^2, q^2)}{2} + (qP)\beta_\pi cos\theta_\pi] \\
&- \beta_\pi\beta_e (q^2 s_\pi)^{\frac{1}{2}} \sin\theta_e \sin\theta_\pi \cos\varphi
\end{aligned}
$$

$$
\begin{aligned}
\beta_\pi &= \frac{\lambda^{\frac{1}{2}}(s_\pi, m^2, m_0^2)}{s_\pi}; \qquad \beta_e = \sqrt{1 - \frac{4m_e^2}{s_e}}; \\
\lambda(x,y,z) &= x^2 + y^2 + z^2 - 2xy - 2xz - 2yz
\end{aligned}
$$

The dilepton center of mass system

We obtained the decay width in the dilepton center of mass system ($\vec{q} = \vec{k}_1 + \vec{k}_2 = 0$), which essentially simplifies calculations. Dividing the pions momenta into longitudinal and transverse parts and using the Lorentz transformations, we express them in terms of the pion momentum p^* in the dipion c.m.s:

$$\begin{aligned} p_{1L} &= \gamma p^* \cos\theta + \beta E_1^*, \\ p_{2L} &= -\gamma p^* \cos\theta + \beta E_2^*, \\ |\vec{p_{1\perp}}| &= |\vec{p_{2\perp}}| = p^* \sin\theta \end{aligned}$$

where θ is the angle between the charged pion in the dipion c.m.s and the dipion flight direction, $\gamma = \frac{M_K^2 - s_\pi - s_e}{2\sqrt{s_\pi s_e}}$ is the relevant Lorentz factor and $\beta = \sqrt{\gamma^2 - 1}$.

Gathering the appropriate expressions, we obtain:

$$
\begin{aligned}
d\Gamma &= \frac{\alpha^2}{4(4\pi)^3 M_K s_e}(|F_1|^2 \vec{p_1}^{\,2} + |F_2|^2 \vec{p_2}^{\,2} + 2(\vec{p_1}\cdot\vec{p_2}) Re(F_1 F_2^*) \\
&+ s_e[\vec{p_1}^{\,2}\vec{p_2}^{\,2} - (\vec{p_1}\cdot\vec{p_2})^2]|F_3|^2)(1 - \frac{v^3}{3}) ds_\pi ds_e dcos\theta; \\
F_1^B &= \frac{2i(\gamma E_2^* - \beta p^* cos\theta)}{(\gamma E_1^* + \beta p^* cos\theta + \omega/2)(M_K^2 - s_\pi)} |M(K^\pm \to \pi^\pm \pi^0)| e^{i\delta_0^2}; \\
F_2^B &= \frac{2i}{(M_K^2 - s_\pi)} |M(K^\pm \to \pi^\pm \pi^0)| e^{i\delta_0^2}; \\
F_1^{DE} &= \frac{2iG_8}{f_\pi} e^{i\delta_1^1} \left(N_E^0 \omega(\gamma E_2^* - \beta p^* cos\theta) + \frac{2}{3}\omega^2 N_E^1 + 2q^2 L_9 \right); \\
F_2^{DE} &= \frac{2iG_8}{f_\pi} e^{i\delta_1^1} \left(N_E^0 \omega(\gamma E_1^* + \beta p^* cos\theta) + \frac{1}{3}\omega^2 N_E^2 \right); \\
F_3^{DE} &= \frac{2eG_8}{f_\pi} e^{i\delta_1^1} N_M^0
\end{aligned}
$$

These formulae allows one to calculate the differential decay width using the minimum set of variables s_e, s_π, θ.

Numerical calculations

First of all we calculated the full decay width in the frameworks of the both approaches. The full width of the $K^{\pm} \to \pi^{\pm}\pi^{0}e^{+}e^{-}$ decay is $\Gamma^{full} = 2.231 \times 10^{-22}$ MeV, whereas the Bremsstrahlung contribution gives $\Gamma^{IB} = 2.181 \times 10^{-22}$ MeV. The branching ratio of the decay under consideration is $\mathrm{BR}(K^{+} \to \pi^{+}\pi^{0}e^{+}e^{-}) = 419.33 \times 10^{-8}$. The calculations in two considered above approaches give the same numbers as it must to be.

Later on all computations have been done using dilepton c.m.s which have an obvious advantage in comparison with the kaon rest system - it needs only three integrations for the full decay width calculation instead of five integrals in the kaon rest frame.

Differential decay width

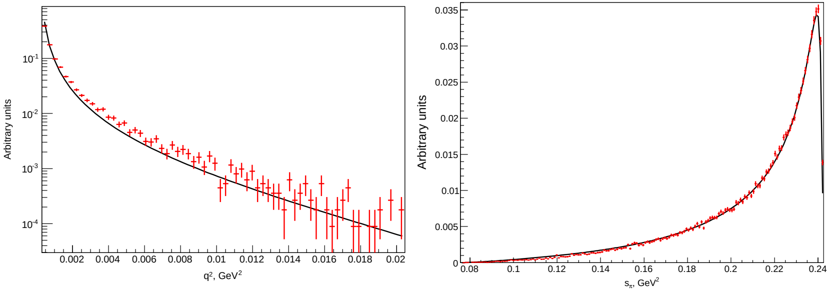

Figure : *Comparison of the full differential decay width with respect to invariants q^2 and S_π, obtained by theoretical calculation in the lepton pair c.m.s (solid curve) and with the MC generator (CERNLIB) (dots are given with their statistical errors).*

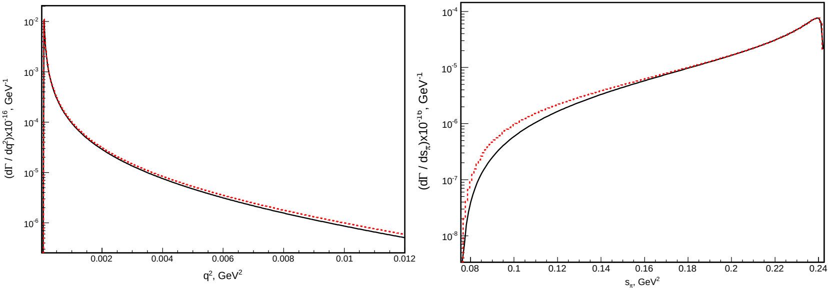

Figure : *Comparison between the decay width of IB contribution (solid line) and the full decay width (dashed line) with respect to the invariant masses of the dilepton and dipion systems.*

The difference between inner bremsstrahlung(IB) and full decay width (with direct emission(DE)) is small and it is evident at large values of q^2 and in the region of small values of s_π.

The direct emission contribution in the full decay width of the $K^\pm \to \pi^\pm \pi^0 e^+ e^-$ is $\sim 2.3\%$, whereas in the case of the $K^\pm \to \pi^\pm \pi^0 \gamma$ it is $\sim 3.32\%$

Summary

The general expression for the differential width of the $K^\pm \to \pi^\pm \pi^0 e^+ e^-$ decay has been investigated in the kaon rest frame and the dilepton c.m.s. Previously we have calculated the differential decay width in terms of the Cabibbo-Maksymowicz variables. We have also used the decay amplitude in the c.m.s of the lepton pair, which is more convenient for computations. By means of these expressions, we have calculated the branching ratio of the $K^\pm \to \pi^\pm \pi^0 e^+ e^-$ channel and obtained the dependencies of the differential width on virtual photon mass q^2 and the invariant mass of pion pair s_π for inner Bremsstrahlung and full decay widths. The comparison between the discussed approaches is presented by using the dependence of the decay width on the invariant masses s_π and q^2.

SG,M.Misheva Eur.Phys.J.C74,2860 (2014)