Extracting Excited Mesons from the Finite Volume

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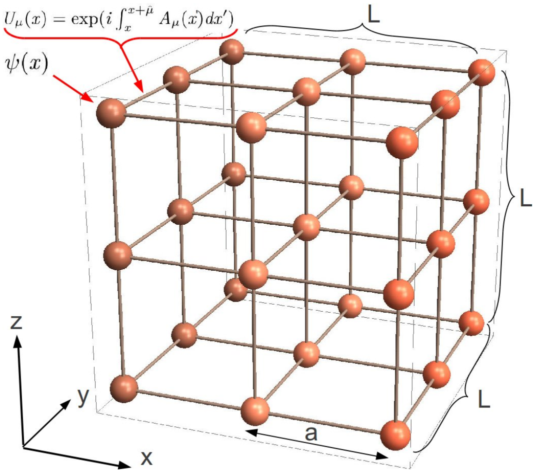
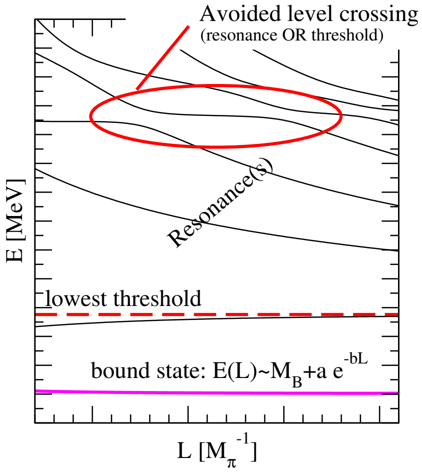
The cubic lattice



- Side length L, $V = L^3$ $(+L_t)$, periodic boundary conditions
  $\Psi(x) \stackrel{!}{=} \Psi(x + \hat{\mathbf{e}}_i\, L)$
  $\rightarrow$ finite volume effects
  $\rightarrow$ Infinite volume $L \rightarrow \infty$ extrapolation
- Lattice spacing a
  $\rightarrow$ finite size effects
  Modern lattice calculations:
  $a \simeq 0.07$ fm $\rightarrow p \sim 2.8$ GeV
  $\rightarrow$ (much) larger than typical hadronic scales;

  not considered here.
- Unphysically large quark/hadron masses
  $\rightarrow$ chiral extrapolation required.

Eigenvalues in the finite volume



Lüscher equation

- Periodic boundary conditions

$$\Psi(x) \overset{!}{=} \Psi(x + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(x) \quad \Rightarrow \quad q_i = \frac{2\pi}{L}\, n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3$$

- Integrals $\to$ Sums

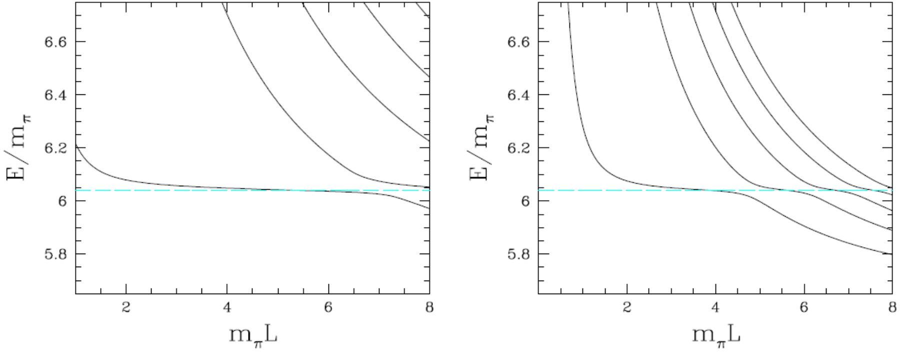
$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}\,|^2) \to \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L}\, \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

- Lüscher equation

$$\boxed{p \cot \delta(p) = -8\pi\, E\, \left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)}$$

  - p: c.m. momentum
  - E: scattering energy
  - $\tilde{G} - \mathrm{Re}\,G$: known kinematical function
  - $\to$ one phase at one energy.

Moving frames to get more levels ($\Sigma^* \to \pi\Lambda$)



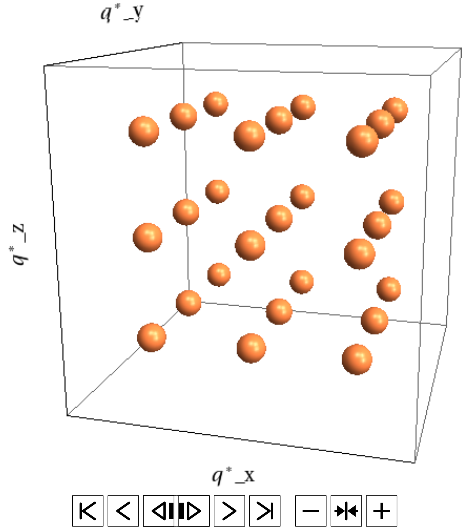
$\boldsymbol{P} = 0$

$\boldsymbol{P} = \frac{2\pi}{L}(0,0,1)$

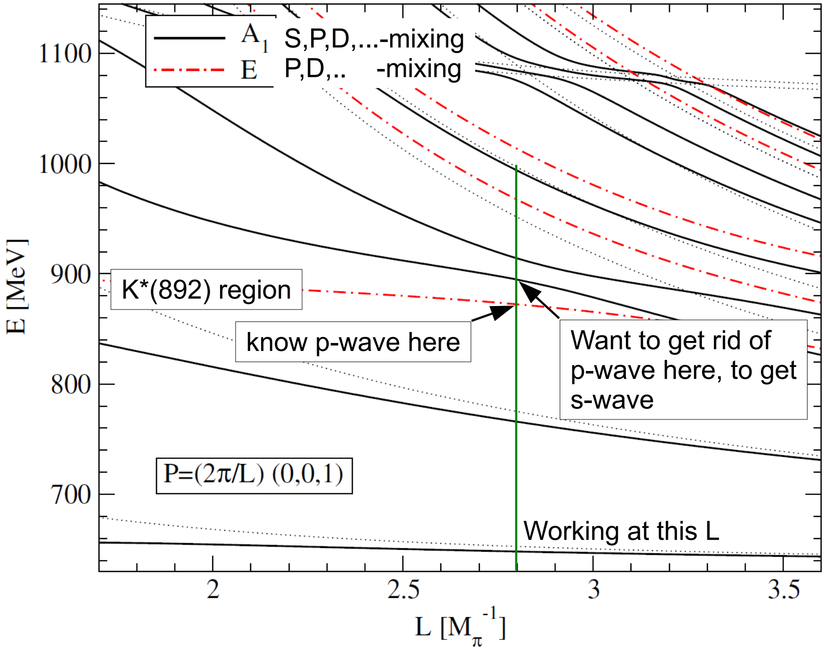
Operators with non-zero momentum of the center-of-mass: $\vec{P} = \vec{p}_1 + \vec{p}_2 \neq 0$
Rummukainen, Gottlieb, NPB (1995)

Breaking of cubic symmetry through boost

Example: Lattice points $\vec{q}^*$ boosted with $P = (0,0,0) \to \frac{2\pi}{L}\,(0,0,2)$:



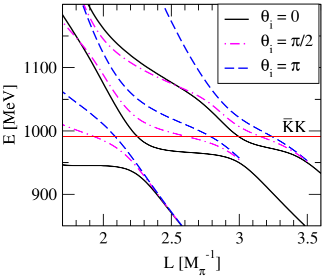
Need for an interpolation in energy ($K\pi$ scattering)



More need for an interpolation in energy (coupled channels)

Twisting the boundary conditions [Bernard, Lage, Meißner, Rusetsky, JHEP (2011), M.D., Meißner, Oset, Rusetsky, EPJA (2011)]

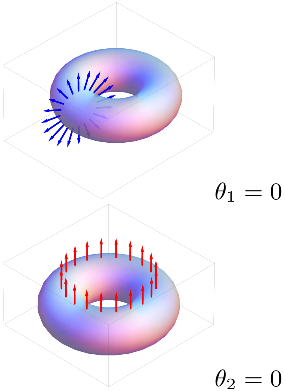
- S-wave, coupled-channels $\pi\pi$, $\bar{K}K \to f_0(980)$.
- Three unknown transitions
  - $V(\pi\pi \to \pi\pi)$
  - $V(\pi\pi \to \bar{K}K)$
  - $V(\bar{K}K \to \bar{K}K)$



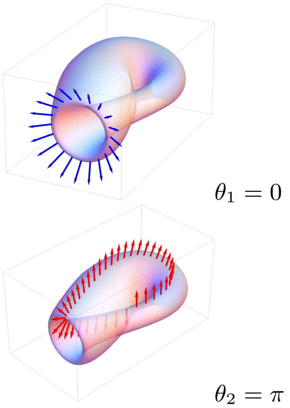
- Twisted B.C. for the s-quark:

$$u(\vec{x} + \hat{\mathbf{e}}_i L) = u(\vec{x})$$
$$d(\vec{x} + \hat{\mathbf{e}}_i L) = d(\vec{x})$$
$$s(\vec{x} + \hat{\mathbf{e}}_i L) = e^{i\theta_i} s(\vec{x})$$

- Periodic B.C.: $\Psi(\vec{x} + \hat{\mathbf{e}}_i L) = \Psi(\vec{x})$
- Periodic in 2 dim.:



- Twisted B.C.: $\Psi(\vec{x} + \hat{\mathbf{e}}_i L) = e^{i\theta_i}\Psi(\vec{x})$
- Periodic/antiperiodic:



Energy interpolation through unitarized ChPT

[M.D., Meißner, JHEP (2012)] using IAM [Oller, Oset, Peláez, PRC (1999)]

Unitary extension of ChPT, can be matched to ChPT order-by-order.

Table: Fitted values for the L_i [$\times 10^{-3}$] and $q_{\max}$ [MeV].

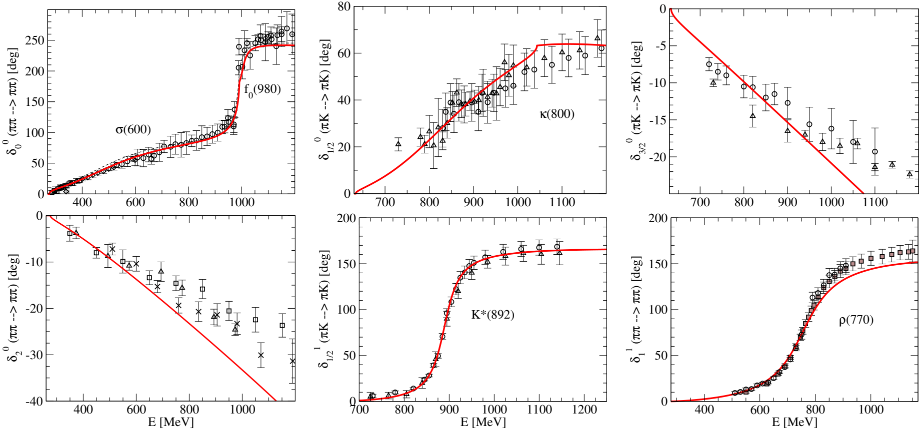
L_1 $0.873^{+0.017}_{-0.028}$	L_2 $0.627^{+0.028}_{-0.014}$	L_3 -3.5 [fixed]	L_4 $-0.710^{+0.022}_{-0.026}$
L_5 $2.937^{+0.048}_{-0.094}$	$L_6 + L_8$ $1.386^{+0.026}_{-0.050}$	L_7 $0.749^{+0.106}_{-0.074}$	$q_{\max}$ [MeV] 981 [fixed]

- A resonance is characterized by its (complex) pole position and residues, corresponding to resonance mass, width, and branching ratio.

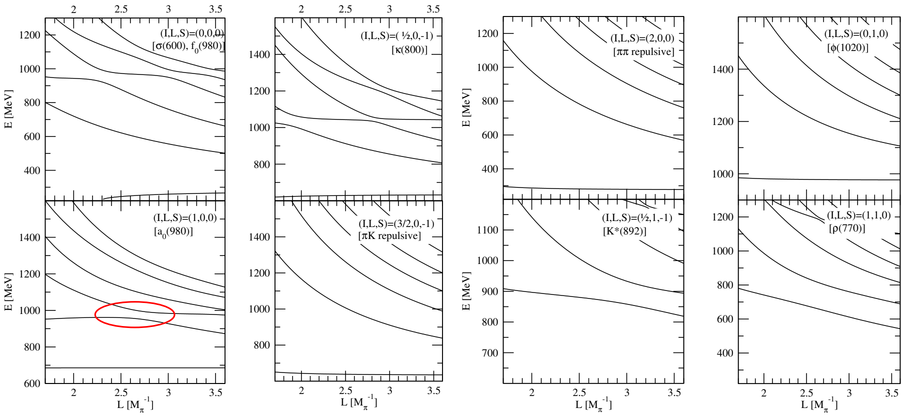
Table: Pole positions z_0 [MeV] and residues $a_{-1}[M_\pi]$ in different channels.
I, L, S: isospin, angular momentum, strangeness.

I	L	S	Resonance	sheet	z_0 [MeV]	a_{-1} [M_π]	a_{-1} [M_π]
0	0	0	$\sigma(600)$	pu	$434+i\,261$	$-31-i\,19\,(\bar{K}K)$	$-30+i\,86\,(\pi\pi)$
0	0	0	$f_0(980)$	pu	$1003+i\,15$	$16-i\,79\,(\bar{K}K)$	$-12+i\,4\,(\pi\pi)$
1/2	0	-1	$\kappa(800)$	pu	$815+i\,226$	$-36+i\,39\,(\eta K)$	$-30+i\,57\,(\pi K)$
1	0	0	$a_0(980)$	pu	$1019-i\,4$	$-10-i\,107\,(\bar{K}K)$	$21-i\,31\,(\pi\eta)$
0	1	0	$\phi(1020)$	p	$976+i\,0$	$-2+i\,0\,(\bar{K}K)$	—
1/2	1	-1	$K^*(892)$	pu	$889+i\,25$	$-10+i\,0.1\,(\eta K)$	$14+i\,4\,(\pi K)$
1	1	0	$\rho(770)$	pu	$755+i\,95$	$-11+i\,2\,(\bar{K}K)$	$33+i\,17\,(\pi\pi)$

Fit to meson-meson PW data using unitary ChPT with NLO terms
[M.D., Meißner, JHEP (2012)] using IAM [Oller, Oset, Peláez, PRC (1999)]



Prediction of levels (also for $M_\pi \neq M_\pi^{\text{phys.}}$)



[M.D., Meißner, JHEP (2012)]
Loops in t- and u-channel (1-loop calculation): [Albaladejo, Rios, Oller, Roca, arXiv: 1307.5169; Albaladejo, Oller, Oset, Rios, Roca, JHEP (2013)]

Fit potential $[V_2 \equiv V_{\mathrm{LO}}$ known/fixed from f_π, f_K, f_η; $s \equiv E^2]$

$$V^{\mathrm{fit}} = \left(\frac{V_2 - V_4^{\mathrm{fit}}}{V_2^2}\right)^{-1}, \quad V_4^{\mathrm{fit}} = a + b(s - s_0) + c(s - s_0)^2 + d(s - s_0)^3 + \cdots$$

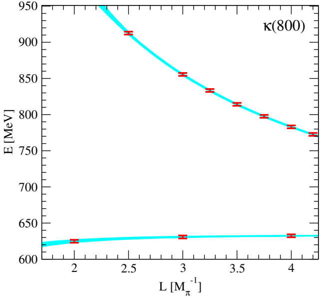


Figure: Pseudo lattice-data and $(s^0,\, s^1,\, s^2)$ fit to those data with uncertainties (bands).

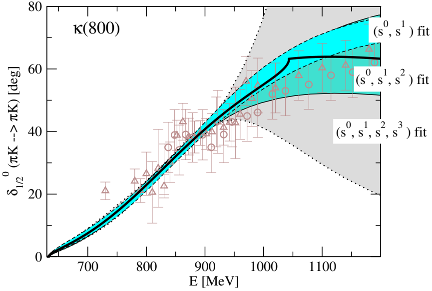
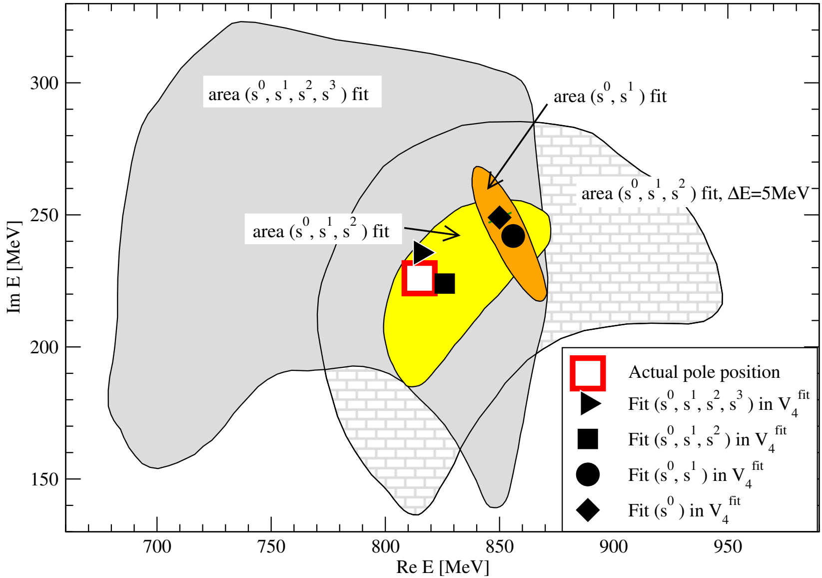


Figure: Solid line: Actual phase shift. Error bands of the $(s^0,\, s^1)$, $(s^0,\, s^1,\, s^2)$, and $(s^0,\, s^1,\, s^2,\, s^3)$ fits.

The $\kappa(800)$ pole



Coupled-channel systems with thresholds

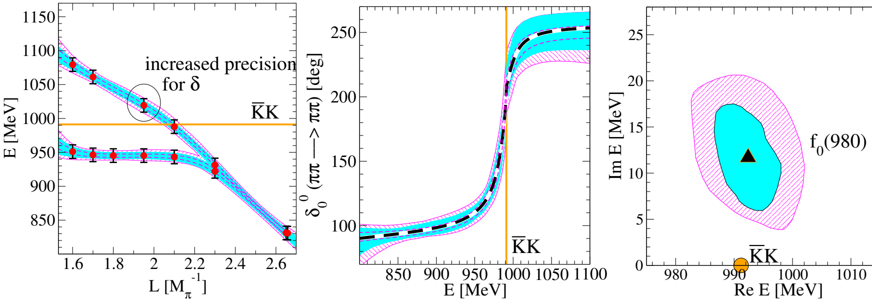
[M.D., Meißner, Oset/Rusetsky, EPJA **47** (2011)]

- Need for an interpolation in energy ($\rightarrow$ Unitarized ChPT,...)
- Expand a two-channel transition V in energy (i,j: $\pi\pi$, $\bar{K}K$):

$$V_{ij}(E) = a_{ij} + b_{ij}(E^2 - 4M_K^2)$$

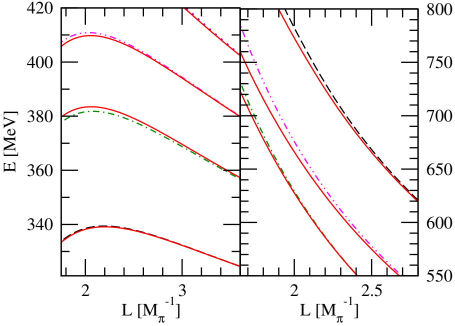
- Include model-independently known LO contribution in a, b.
- Or even NLO contributions (7 LECs: more fit parameters).

lattice data & fit | extracted phase shift | $f_0(980)$ pole position



Mixing of partial waves in boosted multiple channels: $\sigma(600)$

[M.D., E. Oset, A. Rusetsky, EPJA (2012)]



Solid: Levels from A_1^+.

Non-solid: Neglecting the D-wave.

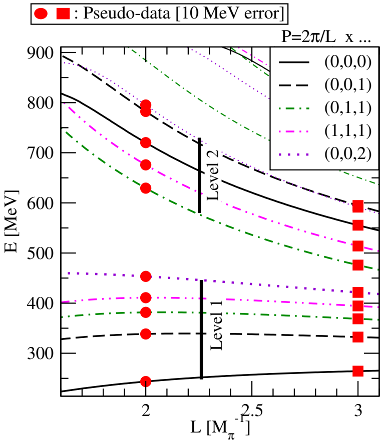
- $\pi\pi$ & $\bar{K}K$ in S-wave, $\pi\pi$ in D-wave.
- Organization in Matrices (A_1^+), e.g. $\vec{P} = (2\pi/L)(0,0,1)$, $(2\pi/L)(1,1,1)$, and $(2\pi/L)(0,0,2)$:

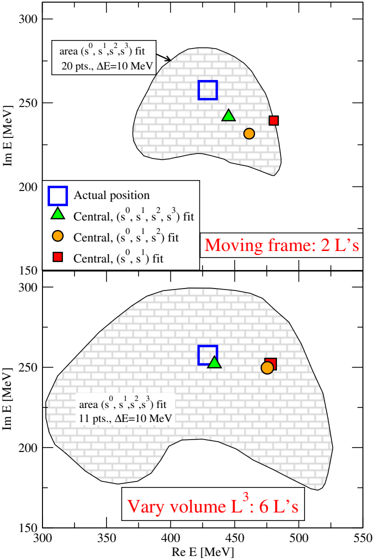
$$V = \begin{pmatrix} V_S^{(11)} & V_S^{(12)} & 0 \\ V_S^{(21)} & V_S^{(22)} & 0 \\ 0 & 0 & V_D^{(22)} \end{pmatrix}$$

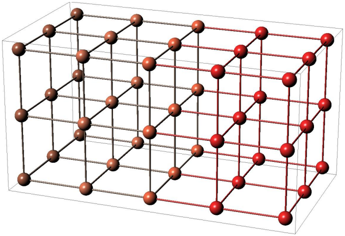
$$\tilde{G} = \begin{pmatrix} \tilde{G}_{00,00}^{R\,(1)} & 0 & 0 \\ 0 & \tilde{G}_{00,00}^{R\,(2)} & \tilde{G}_{00,20}^{R\,(2)} \\ 0 & \tilde{G}_{20,00}^{R\,(2)} & \tilde{G}_{20,20}^{R\,(2)} \end{pmatrix}$$

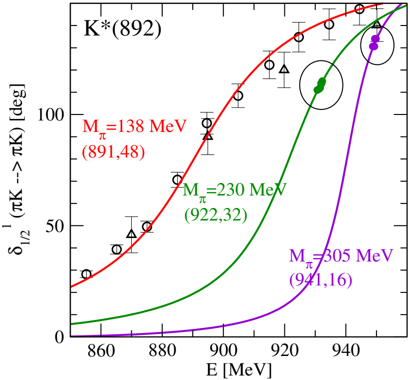
- Phase extraction: Expand and fit V_S, V_D *simultaneously* to different representations, as in case of multi-channels (reduction of error).

- The first two levels for the first five boosts:



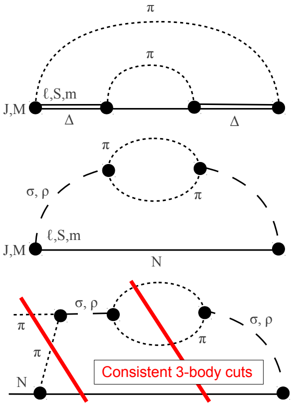






- $L_x = L, L_y = L, L_z = x\,L$
  $x = 1,\ 1.26,\ 2.04$
- $\frac{L}{2\pi}\,\vec{P} = (0,0,0),\ (0,0,1)$

$\rightarrow$ Resonance not covered by eigenlevels.
$\rightarrow$ Find other boosts/spatial setups.



- πN scattering: Known large inelasticities $\pi\pi N$ [$\pi\Delta$, σN, ρN,. . .]
- $\pi\pi/\pi N$ boosted subsystems.
- Is it enough to include (boosted) 2-particle subsystems in the propagator? No.
- Three-body s-channel dynamics requires particle exchange transitions. $\Rightarrow$ Three-body unitarity

[Aaron, Almado, Young, PR 174 (1968) 2022,
Aitchison, Brehm, PLB 84 (1979) 349, PRD 25 (1982) 3069;
Hansen, Sharpe, Davoudi, Briceño. . .]

Perspectives

- Rapid progress in the actual ab-initio calculations of resonances/phase shifts: $\rho(770), a_0(980), K^*$(s, p, d)$, N(1535), N(1650), \Delta(1232), \ldots$.
- Close to the physical point, finite volume effects dominate the spectrum.
- Use finite volume effects in your favor: Lüscher & extensions (coupled channels, moving frames, twisted boundary conditions,...)
- Energy interpolation needed in many aspects —Unitarized ChPT & coupled-channel approaches can provide a framework.
  - Prediction of levels & Chiral extrapolation
    $\rightarrow$ find suitable lattice setups to cover resonance region with eigenstates.
  - provide maximal precision of extracted data.
  - Analysis of lattice data.

- Thank you to the Organizers!
- Thank you for slides: R. Briceño, G. Engel, C. Lang, B. Menadue, M. Petschlies, A. Rusetsky, G. Schierholz, M. Wagner, D. Wilson.

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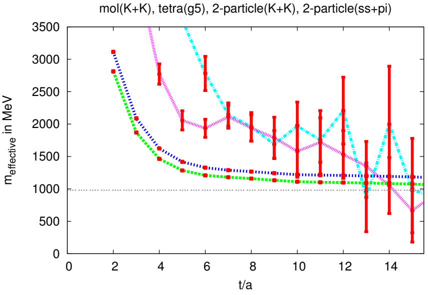
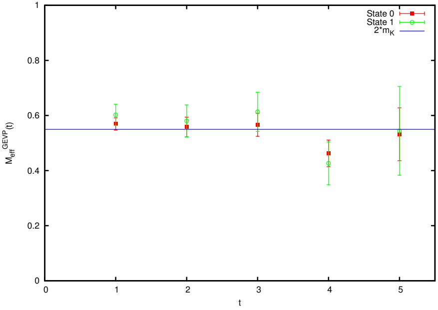
The $a_0(980)$

[Wagner, Daldrop, Abdel-Rehim, Urbach *et. al.* [ETMC], JHEP (2013) & new results]





- $M_\pi \sim 300$ MeV, no singly disconnected diagrams.
- Operators: $\bar{K}K$ molecular, diquark-antidiquark, meson-meson.
- Two low-lying states, large overlap with meson-meson.

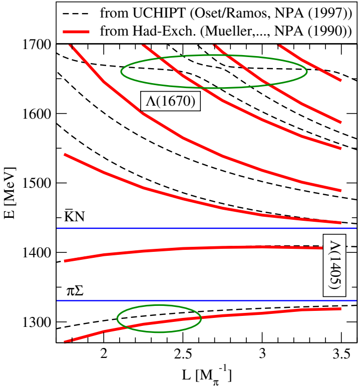
- $M_\pi \sim 300$ MeV, singly disconnected diagrams included.
- Operators: $q\bar{q}$, $\bar{K}K$ molecular.
- Again, two low lying states, no information on additional state.

The $\Lambda(1405)$

[M. D./Haidenbauer/Meißner/Rusetsky, EPJA **47** (2011)]

- (Non-factorizing/off-shell) Lippman-Schwinger equation in the finite volume,

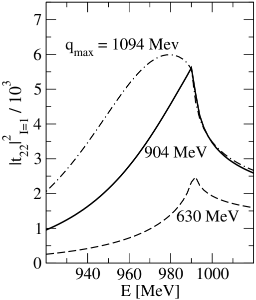
$$T^{(\mathrm{P})}(q'', q') = V(q'', q') + \frac{2\pi^2}{L^3} \sum_{i=0}^{\infty} \vartheta^{(\mathrm{P})}(i) \frac{V(q'', q_i)\, T^{(\mathrm{P})}(q_i, q')}{\sqrt{s} - E_a(q_i) - E_b(q_i)}, \quad q_i = \frac{2\pi}{L}\sqrt{i}\,.$$

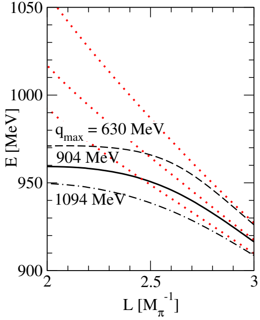


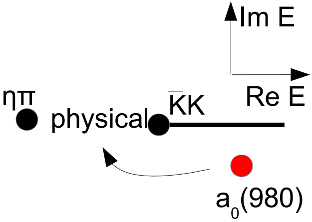
- Access to sub-$\bar{K}N$-threshold dynamics:
- Discrepancies of lowest levels: levels sensitive to different $\Lambda(1405)$ dynamics.
- One- or two-pole structure:
  - Will NOT lead to additional level.
  - but shifted threshold levels.

The $a_0(980)$ in a multi-channel environment

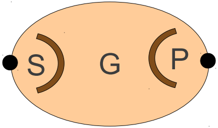
M.D., Meißner, Oset, Rusetsky, EPJA (2011); see also Lage, Meißner, Rusetsky, PLB (2009)







- Infinite volume limit: Rotational symmetry



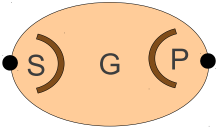
$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}|)\; Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim \delta_{\ell\ell'}\delta_{mm'}\,.$$

S→S	0	0	0
0	$P_{-1} \to P_{-1}$	0	0
0	0	$P_0 \to P_0$	0
0	0	0	$P_1 \to P_1$

Mixing of partial waves

Example: S- and P-waves

- Infinite volume limit: Rotational symmetry



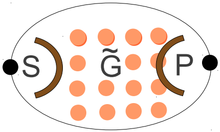
$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, g(|\vec{q}|)\; Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim \delta_{\ell\ell'}\delta_{mm'}\,.$$

- Wigner-Eckart theorem:

S→S	0	0	0
0	P_{-1} → [illegible]	0	0
0	0	[illegible]	0
0	0	0	[illegible] → P_1

Equal

- Finite volume: Rotational symmetry $\to$ Cubic symmetry

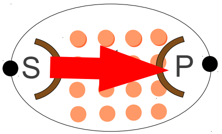


$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' mm'}\,.$$

- $S-G$-wave mixing, but $S-P$ waves still orthogonal:

S$\to$S	0	0	0
0	$P_{-1} \to P_{-1}$	0	0
0	0	$P_0 \to P_0$	0
0	0	0	$P_1 \to P_1$

- Finite volume & boost: Cubic symmetry $\rightarrow$ subgroups of cubic symmetry

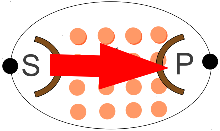


$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' m m'}.$$

- For boost $P = \frac{2\pi}{L}$ (0,1,1):

S$\rightarrow$S	0	S$\rightarrow$P$_0$	0
0	P$_{-1}$ $\rightarrow$ P$_{-1}$	0	P$_{-1}$ $\rightarrow$ P$_1$
P$_0$$\rightarrow$S	0	P$_0$ $\rightarrow$ P$_0$	0
0	P$_1$ $\rightarrow$ P$_{-1}$	0	P$_1$ $\rightarrow$ P$_1$

- Finite volume & boost: Cubic symmetry $\rightarrow$ subgroups of cubic symmetry



$$\frac{1}{L^3}\sum_{\vec{n}} g(|\vec{q}|)\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell' m'}(\theta,\phi) \sim A_{\ell\ell' m m'}\,.$$

- More complicated boosts:

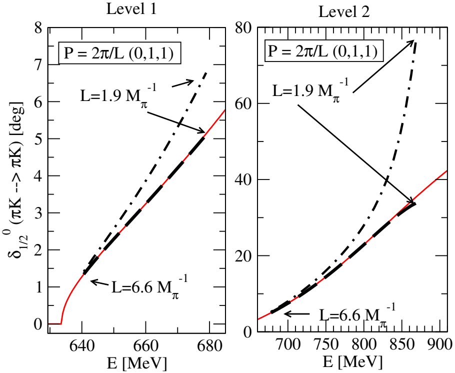
$S \rightarrow S$	$S \rightarrow P_{-1}$	$S \rightarrow P_0$	$S \rightarrow P_1$
$P_{-1} \rightarrow S$	$P_{-1} \rightarrow P_{-1}$	$P_{-1} \rightarrow P_0$	$P_{-1} \rightarrow P_1$
$P_0 \rightarrow S$	$P_0 \rightarrow P_{-1}$	$P_0 \rightarrow P_0$	$P_0 \rightarrow P_1$
$P_1 \rightarrow S$	$P_1 \rightarrow P_{-1}$	$P_1 \rightarrow P_0$	$P_1 \rightarrow P_1$

Disentanglement of partial waves [M.D., Meißner, Oset, Rusetsky, EPJA (2012)]

Example: S- and P-waves for the $\kappa(800)/K^*(892)$ system

Knowledge of P-wave (from separate analysis of lattice data) allows to disentangle the S-wave:

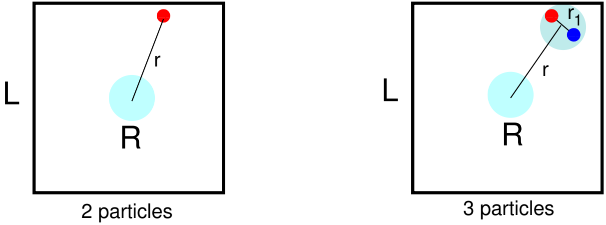
$$p\,\cot\delta_S = -8\pi\,E\,\frac{p\,\cot\delta_P\,\hat{G}_{SS} - 8\pi E(\hat{G}_{SP}^2 - \hat{G}_{SS}\hat{G}_{PP})}{p\,\cot\delta_P + 8\pi\,E\,\hat{G}_{PP}}$$



- $\delta_S \equiv \delta^0_{1/2}(\pi K \to \pi K)$
- Red solid: Actual S-wave phase shift.
- Dash-dotted: Reconstructed S-wave phase shift, PW-mixing ignored.
- Dashed: Reconstructed S-wave phase shift, PW-mixing disentangled.
- small p-wave: Level shift $\Delta E \simeq -\dfrac{6\pi E_S \delta_P}{L^3 p\omega_1\omega_2}$

Rummukainen, Gottlieb, NPB (1995); Kim, Sachrajda, Sharpe, NPB (2005); Davoudi, Savage, PRD (2011), Z. Fu, PRD (2012); Leskovec, Prelovsek, PRD (2012); Dudek, Edwards, Thomas, PRD (2012); Hansen, Sharpe, PRD 86 (2012); Briceño, Davoudi, arXiv:1204.1110; Göckeler, Horsley, Lage, Meißner, Rakow, Rusetsky, Schierholz, Zanotti, PRD (2012)

Three particles in a finite volume



- In case of 2 particles: $r \gg R$, when particles are near the walls
- In case of 3 particles: it may happen that $r \gg R, \quad r_1 \simeq R$, when the particles are near the walls

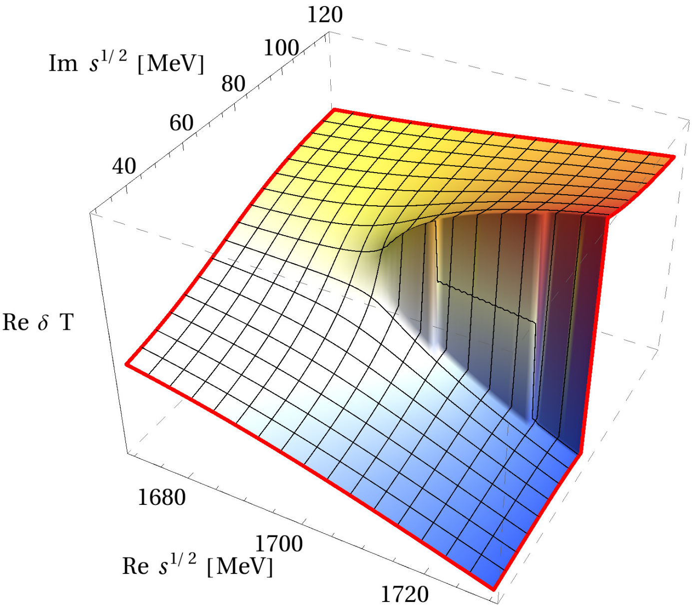
The problem with the disconnected contributions: is the finite-volume spectrum in the 3-particle case determined solely through the on-shell scattering matrix?

- Despite the presence of the disconnected contributions, the energy spectrum of the 3-particle system in a finite box is still determined by the <u>on-shell</u> scattering matrix elements in the infinite volume

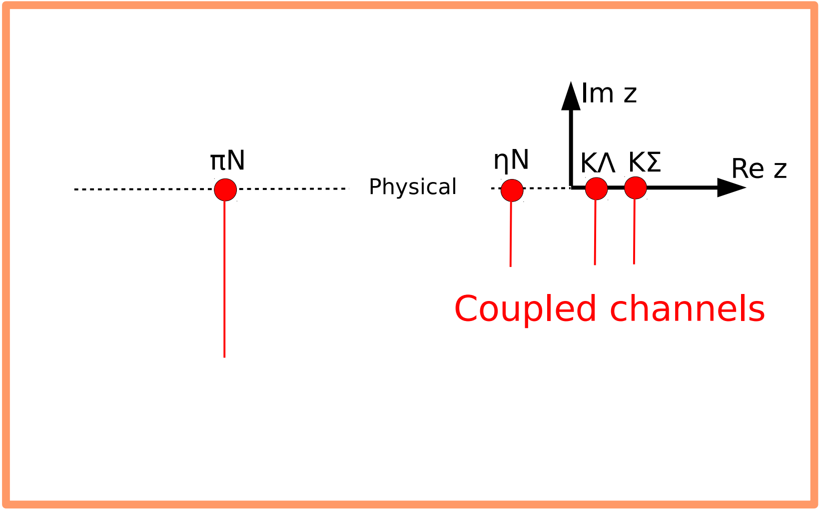
[Polejaeva, Rusetski, EPJA (2012)]

Three particles: Threshold openings in the complex plane

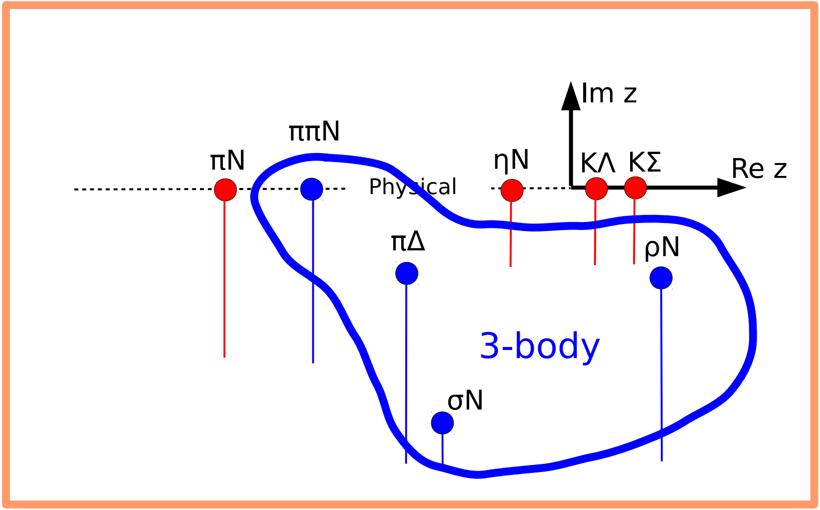
Existence shown model-independently in [S. Ceci, M.D., C. Hanhart *et. al.*, PRC 84 (2011)]



$(z = E)$



$(z = E)$

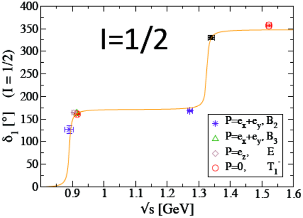


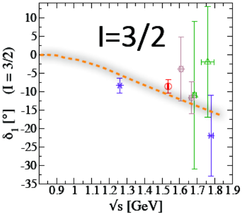
Kπ scattering and K* width in moving frames

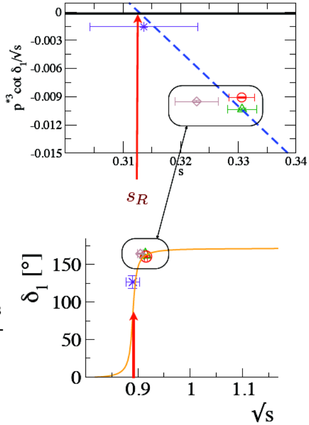
Prelovsek, Leskovec, Lang, Mohler,
this conf. and *arXiv: 1307.0736*

p-wave, coupled system of 5 $q\bar{q}$ and 3 Kπ operators,
total momentum P=(000),(001),(011)

Representations B_2, B_3, E, T_1^-



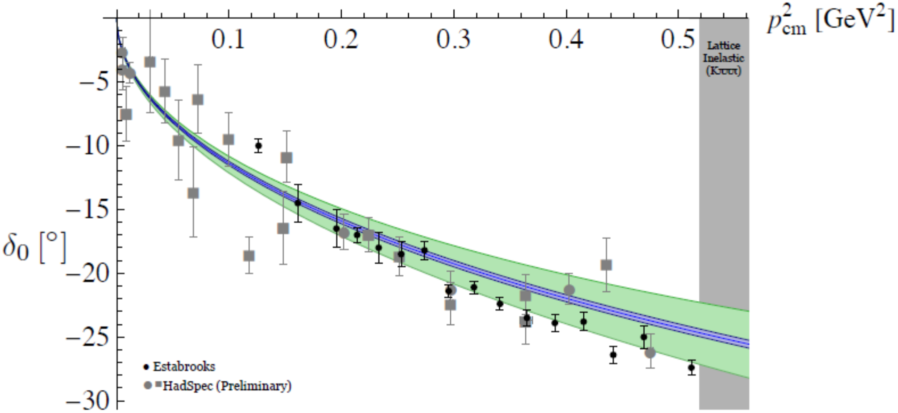




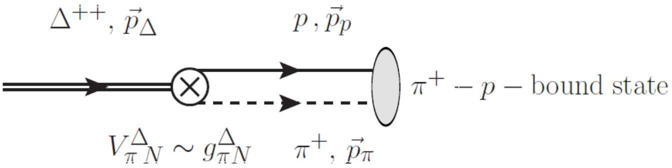
$$\frac{p^{*3}}{\sqrt{s}}\cot\delta_1(s) = \left[\sum_{K^*}\frac{g^2_{K_i^*}}{6\pi}\frac{1}{m^2_{K_i^*}-s}\right]^{-1} \qquad \Gamma[K^*\to K\pi] = \frac{g^2}{6\pi}\frac{p^{*3}}{s}$$

	$m_{K^*(892)}$ [MeV]	$g_{K^*(892)}$ [no unit]	$m_{K^*(1410)}$ [GeV]	$g_{K^*(1410)}$ [no unit]
lat	891 ± 14	5.7 ± 1.6	1.33 ± 0.02	input
exp	891.66 ± 0.26	5.72 ± 0.06	1.414 ± 0.0015	1.59 ± 0.03

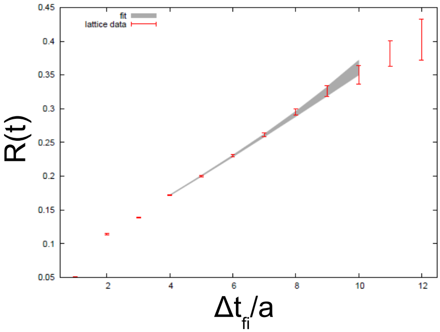
D. Wilson [HadSpec], Lattice 2013.



Δ resonance from the transfer matrix method [ETMC, arXiv:1305.6081]



$$R(\Delta t_{fi}, \vec{Q}, \vec{q}) = \frac{C_\mu^{\Delta\to\pi N}(\Delta t_{fi}, \vec{Q}, \vec{q})}{\sqrt{C_\mu^{\Delta}(\Delta t_{fi}, \vec{Q})\, C^{\pi N}(\Delta t_{fi}, \vec{Q}, \vec{q})}} \sim \langle \Delta | H | \pi N \rangle \Delta t_{fi}$$



$$\langle \Delta | H | \pi N \rangle \rightarrow g^{\Delta}_{\pi N}$$

$$g^{\Delta}_{\pi N}(\mathrm{lat}) = 27.0 \pm 0.6 \pm 1.5$$

Rakow/QCDSF [Lattice 2013]:
$\pi^+ p \rightarrow K^+ \Sigma^+$: σ~0.4 (0.2) fm
at SU(3) symmetrical point