Role of the $a_0(980)$, $f_0(980)$ resonances in $\eta \to 3\pi$ from the Khuri-Treiman formalism

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Abstract. The chiral expansion of the $\eta \to 3\pi$ decay amplitude at NLO fails to reproduce the recent high accuracy measurements of the Dalitz plot parameters. We reconsider the idea of employing the chiral expansion in an unphysical region surrounding the Adler zero and then deducing the amplitude in the physical region using the Khuri-Treiman dispersive formalism. We further present an extension of this formalism which takes into account $K\bar{K}$ inelastic rescattering effects and thus provides a realistic description of the double $a_0(980)$, $f_0(980)$ resonance effect. We evaluate how the influence of these resonances propagates down to the low-energy decay region and show that it is significant, in particular for the $\eta \to 3\pi^0$ decay, and improve the agreement of the predicted Dalitz plot parameters with experiment.

1 Introduction

The fact that the masses of the u, d, s quarks are much smaller than 1 GeV has an essential influence on the physics of light mesons (indeed if these masses where $O(1)$ GeV the lightest meson in QCD would be the 0^{++} glueball). In this context, $\eta \to 3\pi$ decays are key processes for the determination of the $u - d$ quark mass difference. At next-to-leading order (NLO) in the low-energy chiral expansion, the amplitude can be expressed in a very predictive form involving the double quark mass ratio

$$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_d + m_u)^2/4} \tag{1}$$

as an overall multiplicative factor [1, 2]. Unfortunately, the NLO amplitude fails to reproduce the experimental values of the Dalitz plot parameters which have been measured quite precisely (see [3, 4] and references therein), which indicates that higher order chiral effects must be accounted for.

A calculation of the complete NNLO amplitude has been performed [5] but our present knowledge of the $O(p^6)$ couplings is not sufficient for making model independent predictions. It is plausible, however, that the important $O(p^6)$ effects are those associated with the restoration of unitarity, or final-state interactions (FSI), effects. This has led to the suggestion of relying on the chiral expansion in the subthreshold unphysical region only, where these effects are suppressed, and construct the physical amplitude from an extrapolation based on exact unitarity and analyticity [6, 7]. A dispersive framework for $\eta \to 3\pi$ was developed in refs. [2, 8] based on the classic work of Khuri and Treiman [9]. In

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this formalism, $\pi\pi$ rescattering is treated in the elastic approximation. We present here an extension which accounts for the leading inelastic effects in $\pi\pi$ scattering and also includes $\eta\pi$ rescattering such that a realistic description of the role of the two light resonances $a_0(980)$, $f_0(980)$ in the dispersive integrals can be achieved.

2 The Khuri-Treiman equations with elastic unitarity

We restrict ourselves to regions of the Mandelstam plane where the imaginary parts of the $\eta\pi \to \pi\pi$ partial-waves with $J \geq 2$ can be neglected compared to those with $J = 0, 1$ (i.e. $|s|, |t|, |u| \lesssim 1$ GeV^2). It can then be shown [2, 8] that the η decay amplitudes can be expressed in terms of three functions of a single variable,

$$\begin{aligned}
\mathcal{T}_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) =& -\epsilon_L \Big[M_0(s) - \tfrac{2}{3}M_2(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u)\Big] \\
\mathcal{T}_{\eta\to 3\pi^0}(s,t,u) =& -\epsilon_L \Big[M_0(s) + M_0(t) + M_0(u) + \tfrac{4}{3}(M_2(s) + M_2(t) + M_2(u))\Big]
\end{aligned} \tag{2}$$

where the Mandelstam variables (in the charged case) are given by $s = (p_{\pi^+} + p_{\pi^-})^2$, $t = (p_{\pi^-} + p_{\pi^0})^2$, $u = (p_{\pi^+} + p_{\pi^0})^2$ and the overall factor $\epsilon_L = Q^{-2}\, m_K^2(m_K^2 - m_\pi^2)/3\sqrt{3}F_\pi^2 m_\pi^2$. The functions $M_I(w)$, furthermore, are analytic in the whole complex plane except for a cut along the positive real axis $w \geq 4m_\pi^2$. From eq. (2), it is easy to derive the form of the three partial-waves with $J = 0, 1$ corresponding to a definite isospin, I, of one of the $\pi\pi$ pairs

$$\mathcal{M}_J^I(s) = \lambda^{(I)}\, \frac{\epsilon_L}{16\pi} \kappa^J(s)(M_I(s) + \hat{M}_I(s)) \tag{3}$$

where $\kappa^2(s) = (1 - 4m_\pi^2/s)(s - (m_\eta + m_\pi)^2)(s - (m_\eta - m_\pi)^2)$ and the isospin factors are $\lambda^{(0)} = \sqrt{6}/2$, $\lambda^{(1)} = 1/3$, $\lambda^{(2)} = -1$. The functions $\hat{M}_I$ can be expressed as simple integrals of the functions M_I (see [2]). They have left-hand cuts which extend into the complex plane and approach infinitesimally close to the unitarity cut but remain well separated from it.

Assuming elastic unitarity for the three partial waves

$$\mathrm{Im}\,[\mathcal{M}_J^I(s)] = \sigma_{\pi\pi}(s)(f_J^I(s))^*\, \mathcal{M}_J^I(s) \tag{4}$$

(where f_J^I is a $\pi\pi \to \pi\pi$ partial-wave) one can derive the discontinuities of the functions M_I and then, using dispersive representations together with the Omnès method[10] arrive at the following set of coupled integral equations for the functions M_I [2]

$$\begin{aligned}
M_0(w) &= \Omega_0(w)[\alpha_0 + w\beta_0 + w^2\left(\gamma_0 + \hat{I}_0(w)\right)] \\
M_1(w) &= \Omega_1(w)\, w[\beta_1 + \hat{I}_1(w)] \\
M_2(w) &= \Omega_2(w)\, w^2[\hat{I}_2(w)]
\end{aligned} \tag{5}$$

where the functions $\hat{I}_I$ involve integrals over the left-cut functions

$$\hat{I}_I(w) = -\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds'\, \frac{\mathrm{Im}\,[1/\Omega_I(s')]}{(s')^n(s'-w)}\, \hat{M}_I(s') \tag{6}$$

(with $n = 2$ when $I = 0, 2$ and $n = 1$ when $I = 1$) and $\Omega_I(s)$ are the Omnès functions associated with the elastic $I = 0, 2$, $J = 0$ and $I = J = 1$ $\pi\pi$ phase-shifts.

2.1 Matching with $O(p^4)$ ChPT

Dispersive Omnès representations involve a polynomial arbitrariness. The form used above, following ref. [2], is chosen to depend on four polynomial parameters α_0, β_0, γ_0, β_1. This is particularly convenient for the present purpose since all four parameters can be fixed from matching conditions with the NLO amplitude. These conditions must ensure that,

$$\mathcal{T}^{disp}(s,t,u) - \mathcal{T}^{NLO}(s,t,u) = O(p^6)\ . \tag{7}$$

This will be implemented here in a form essentially equivalent to ref. [2], but slightly more symmetric, obtained by expanding the difference in eq. (7) as a polynomial in s, t, u and requiring that the $O(p^4)$ part vanishes.

3 KT equations with inelastic channels

The formalism discussed above does not account properly for the role of the $f_0(980)$ resonance in the dispersive integrals, which induces a strong inelasticity in $I = J = 0$ $\pi\pi$ scattering and also ignores the role of the $a_0(980)$ resonance. In order to improve on that, we must go beyond the elastic approximation and include the $K\bar{K}$ and $\eta\pi$ channels in the unitarity relation. The $I = 0$ S-wave rescattering will then be described by a 2×2 matrix involving the $(\pi\pi)_{I=0}$ and $(K\bar{K})_{I=0}$ channels which we call $\boldsymbol{T}^{(0)}$. Similarly, we will consider a 2×2 matrix related to the $I = 1$ channels $\eta\pi$ and $(K\bar{K})_{I=1}$, which will play a role as an initial-state interaction. We also collect the S-wave isospin violating amplitudes into two matrices

$$\boldsymbol{T}^{(01)} = \begin{pmatrix} (\pi\pi)_0 \to \eta\pi & (K\bar{K})_0 \to \eta\pi \\ (\pi\pi)_0 \to (K\bar{K})_1 & (K\bar{K})_0 \to (K\bar{K})_1 \end{pmatrix}, \quad \boldsymbol{T}^{(12)} = \begin{pmatrix} \eta\pi^+ \to \pi^+\pi^0 \\ K^+\bar{K}^0 \to \pi^+\pi^0 \end{pmatrix} \tag{8}$$

For illustration, the unitarity relation for $\boldsymbol{T}^{(01)}$ reads

$$\mathrm{Im}\,[\boldsymbol{T}^{(01)}] = \boldsymbol{T}^{(0)*}\,\Sigma^0\,\boldsymbol{T}^{(01)} + \boldsymbol{T}^{(01)*}\,\Sigma^1\,\boldsymbol{T}^{(1)} + \boldsymbol{T}^{(0)*}\begin{pmatrix}0 & 0\\ 0 & \Delta\sigma_K\end{pmatrix}\boldsymbol{T}^{(1)} \tag{9}$$

and contains three contributions. In the third one, isospin violation is induced by the physical $K^+ - K^0$ mass difference,

$$\Delta\sigma_K(s) = \frac{1}{2}\left(\theta(s-4m^2_{K^+})\sqrt{1-4m^2_{K^+}/s} - \theta(s-4m^2_{K^0})\sqrt{1-4m^2_{K^0}/s}\right)\ . \tag{10}$$

Correspondingly to this matrix unitarity relation, one can derive a matrix Omnès generalisation of the equation for the amplitude function M_0,

$$\boldsymbol{M}_0(w) = \boldsymbol{\Omega}_0(w)\left[\boldsymbol{P}_0(w) + w^2\left(\hat{\boldsymbol{I}}_A(w) + \hat{\boldsymbol{I}}_B(w)\right)\right]{}^t\boldsymbol{\Omega}_1(w) \tag{11}$$

and a similar one for the M_2 equation. Here $\boldsymbol{\Omega}_I$ are Omnès matrices corresponding to the scattering matrices $\boldsymbol{T}^{(I)}$ and $\boldsymbol{P}_0$ is a matrix of polynomials which will get completely determined, as in the elastic case, from chiral matching conditions. The matrix $\hat{\boldsymbol{I}}_B$, which corresponds to the third term in the unitarity relation (9) reads

$$\hat{\boldsymbol{I}}_B(w) = \frac{32}{\sqrt{6}\,\epsilon_L}\int_{4m^2_{K^+}}^{\infty}\frac{ds'\,\Delta\sigma_K(s')}{(s')^2(s'-w)}\boldsymbol{\Omega}_0^{-1*}(s')\,\boldsymbol{T}^{(0)*}(s')\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\boldsymbol{T}^{(1)}(s')\,{}^t\boldsymbol{\Omega}_1^{-1}(s')\ . \tag{12}$$

Finally, the matrix $\hat{\boldsymbol{I}}_A$ collects integrals over the left-cut pieces of the partial-waves (analogous to eq. (6)).

4 Results and conclusions

Concerning the left-cut parts of the partial-wave amplitudes, we know how to relate $\hat{M}_I$ to the right-cut functions M_I (via the representation (2) of the $\eta \to 3\pi$ amplitude). Similar relations can be derived, in principle, for the $K\bar{K}$ amplitudes (which requires to consider also the cross-channel amplitudes). We adopt here the simple approximation of neglecting the left-cut integrals involving $K\bar{K}$ amplitudes. As a related approximation, we perform the chiral matching of the $K\bar{K}$ amplitudes using the LO expressions (which have no left-hand cut).

We have solved numerically the KT equations using an $I = 0$ T-matrix obtained from simple fits to available experimental data and imposing asymptotic conditions which insure a unique solution for the Omnès matrix. For the $I = 1$ T-matrix, we have used a model constrained by chiral symmetry, at low energy, and by experimental information on the isovector resonances (see ref. [11]).

The results for the Dalitz plot parameters are shown in Table 1. One sees that the KT amplitudes matched to the NLO amplitudes provide a considerable improvement over the NLO amplitudes in the physical region. This is in agreement with the findings of ref. [8]. The table shows that including the effects of the 1 GeV scalar resonances further improves the agreement with experiment. Concerning the quark mass ratio, we find $Q \simeq 21.8$ with the elastic KT amplitudes and essentially the same result with the inelastic ones. The main source of uncertainty for Q arises from the NNLO contributions in the matching equations.

Table 1. $\eta \to 3\pi$ Dalitz plot parameters from the NLO chiral expansion and from elastic and inelastic chirally matched Khuri-Treiman amplitudes.

$\eta \to \pi^+\pi^-\pi^0$	$O(p^4)$	single-ch.	coupled-ch.	KLOE	BESIII
a	-1.328	-1.154	-1.142	-1.095(4)	-1.128(15)
b	0.429	0.202	0.171	0.145(6)	0.153(17)
d	0.089	0.094	0.097	0.081(7)	0.085(16)
f	0.016	0.108	0.123	0.141(10)	0.173(28)
g	-0.081	-0.087	-0.088	-0.044(16)	–
$\eta \to \pi^0\pi^0\pi^0$				PDG	
α	+0.0142	-0.0274	-0.0337	-0.0315(15)	

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