Theoretical analysis of the $\gamma\gamma^{(*)} \to \pi^0\eta$ process

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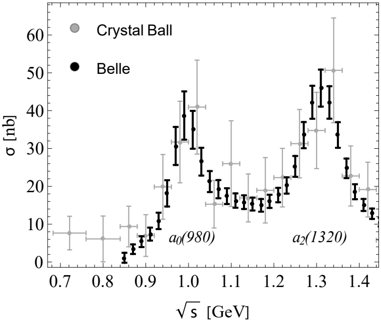
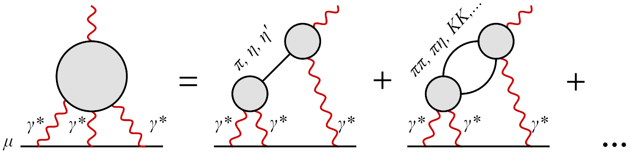
Motivation

- Existing experiments:
  Crystal Ball ('89), Belle ('09)
- Presence of the resonances:
  $a_0(980)$, $J^{PC} = 0^{++}$ and $a_2(1320)$, $J^{PC} = 2^{++}$
- Upcoming experiment:
  BESIII for $\gamma\gamma^* \to \pi^0\eta$ with a spacelike photon
- Input to muon $(g-2)_\mu$





S-matrix constraints

- **Unitarity**: $SS^\dagger = S^\dagger S = \mathbb{1}$

  One can separate the trivial part and obtain

  $$S = \mathbb{1} + iT \quad \Longrightarrow \quad T - T^\dagger = iT^\dagger T = iTT^\dagger$$

  For the scattering amplitude it leads to

  $$T_{fi} - T^*_{if} = 2i\mathrm{Im}\, T_{fi} = i\sum_n \int d\Phi_n T^*_{fn} T_{ni}$$

  Partial wave decomposition:

  $$H_{\lambda_1\lambda_2}(s,t) = \sum_{\text{even } J\geq 0} (2J+1) h_{J,\lambda_1\lambda_2}(s) d^J_{\lambda_1-\lambda_2,0}(\theta), \quad T(s,t) = \sum_{l=0}^{\infty}(2J+1)P_l(\cos\theta)t_l(s)$$

  The **unitarity condition**: $\mathrm{Im}\, h_{J,\lambda_1\lambda_2}(s) = \rho(s) h_{J,\lambda_1\lambda_2}(s) t^*_J(s)$, phase space factor $\rho(s) = \frac{1}{16\pi}\frac{2p_{cm}(s)}{\sqrt{s}}$

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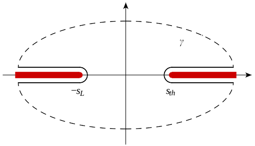
- **Crossing symmetry**

- **Analyticity**: $T(s)_{\text{physical}} = \lim_{\epsilon\to 0+} T(s+i\epsilon)$

  Cauchy's integral formula $f(s) = \frac{1}{2\pi i}\oint_{\gamma_s} \frac{f(s')}{s'-s} ds'$

  Single-channel **dispersion relation**:
  $$t_l(s) = \underbrace{\frac{1}{\pi}\int_{-\infty}^{s_L} \frac{\mathrm{Im}\, t_l(s')}{s'-s} ds'}_{\text{l.h.c.}} + \underbrace{\frac{1}{\pi}\int_{s_{thr}}^{\infty} \frac{\mathrm{Im}\, t_l(s')}{s'-s} ds'}_{\text{r.h.c.}}$$



$$t_l(s) = \frac{N_l(s)}{D_l(s)}, \quad \Omega_l(s) \equiv 1/D(s)$$

Coupled-channel dispersion relation

Now consider the process $\gamma\gamma \to \pi^0\eta$ with intermediate $\pi\eta, K\bar{K}$ states and write down the d.r. for function $\Omega^{-1}(h(s) - h^{\text{Born}}(s))$, which contains both left- and right-hand cuts.

Coupled-channel once-subtracted dispersion relation for $I = 1$ s-wave scattering:

$$\begin{pmatrix} h^1_{0,++}(s) \\ k^1_{0,++}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ k^{1,\text{Born}}_{0,++}(s) \end{pmatrix} + \Omega^1_0(s)\Bigg[\begin{pmatrix} a \\ b \end{pmatrix} + \frac{s - s_{th}}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s' - s_{th}} \frac{\Omega^1_0(s')^{-1}}{s' - s}\begin{pmatrix} \text{Disc}\, h^{1,V}_{0,++}(s') \\ \text{Disc}\, k^{1,V}_{0,++}(s') \end{pmatrix}$$
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Hadronic Omnès matrix is normalized as $\Omega^1_0(s_{th}) = \mathbb{1}$

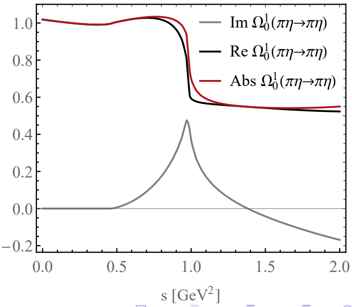
$$\Omega^1_0(s) = \begin{pmatrix} \Omega^1_0(s)_{\pi\eta\to\pi\eta} & \Omega^1_0(s)_{\pi\eta\to K\bar{K}} \\ \Omega^1_0(s)_{K\bar{K}\to\pi\eta} & \Omega^1_0(s)_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

and satisfies the following unitarity condition

$$\mathrm{Disc}\,\Omega^1_0(s) = \frac{1}{2i}(\Omega^1_0(s+i\epsilon) - \Omega^1_0(s-i\epsilon))$$

Omnès matrix $\Longrightarrow$ cutoff scale Λ_S.

Omnès function: I. V. Danilkin, L. I. R. Gil, and M. F. M. Lutz, Phys. Lett. **B703**, 504 (2011).



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Vector mesons

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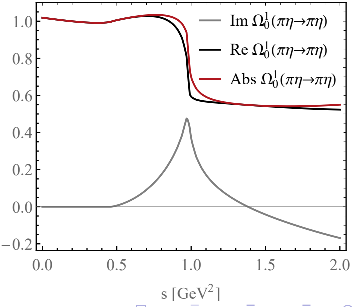
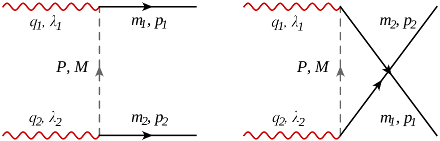


Left-hand cuts



$$\mathscr{L}_{VP\gamma} = e\, C_V\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \partial_\alpha V_\beta$$

C_V are the radiative couplings fixed from the partial widths of light vector mesons

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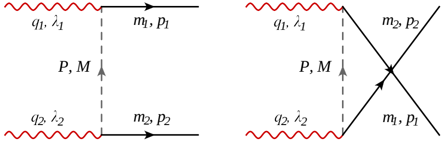

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The helicity amplitudes can be expressed as

$$H_{\lambda_1\lambda_2} = \epsilon_\mu(q_1,\lambda_1)\, \epsilon_\nu(q_2,\lambda_2) \left[F_1(s,t) L_1^{\mu\nu} + F_2(s,t) L_2^{\mu\nu}\right]$$

Lorentz structures are defined with $\Delta = p_1 - p_2$

$$L_1^{\mu\nu} = q_1^\nu q_2^\mu - (q_1\cdot q_2) g^{\mu\nu},$$

$$L_2^{\mu\nu} = (\Delta^2 (q_1\cdot q_2) - 2(q_1\cdot\Delta)(q_2\cdot\Delta)) g^{\mu\nu} - \Delta^2 q_1^\nu q_2^\mu - 2(q_1\cdot q_2)\Delta^\mu \Delta^\nu + 2(q_2\cdot\Delta) q_1^\nu \Delta^\mu + 2(q_1\cdot\Delta) q_2^\mu \Delta^\nu,$$

Invariant amplitudes for the vector meson exchange

$$F_1(s,t) = -\sum_V 2 e^2 C_{12} \left(\frac{t}{t - M_V^2} + \frac{u}{u - M_V^2}\right)$$

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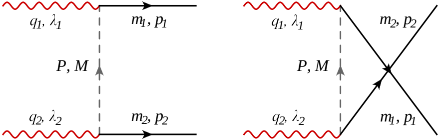

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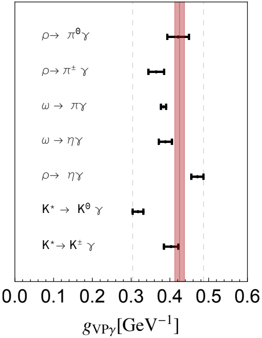

SU(3): $C_{\rho\to\pi^0\gamma} = \frac{1}{\sqrt{3}} C_{\rho\to\pi^0\eta} = \ldots g_{eff}$

Cross channel process $\eta \to \pi^0 \gamma\gamma$

Crossing symmetry implies that the invariant amplitudes $F_{1,2}(s,t)$ describe not only the scattering process $\gamma\gamma \to \pi^0\eta$ but also the decay process $\eta \to \pi^0\gamma\gamma$

$$\frac{d^2\Gamma}{ds\,dt} = \frac{1}{(2\pi)^3}\frac{1}{32\,m_\eta^3}\sum_{\lambda_1,\lambda_2}|H_{\lambda_1\lambda_2}(s,t)|^2 = \frac{1}{(2\pi)^3}\frac{1}{32\,m_\eta^3}\left(\frac{s^2}{2}F_1(s,t)^2 + 8F_2(s,t)\left(m_\eta^2 m_\pi^2 - tu\right)^2\right)$$

where crossing implies the following relations to the decay invariants $s \to M_{\gamma\gamma}^2$ and $t \to M_{\gamma\pi}^2$

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Loop function: $I(s,m^2) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{xy}{m^2 - sxy}$

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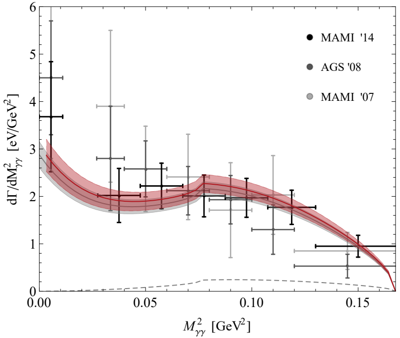
$$\Gamma_{\eta\to\pi^0\gamma\gamma} = 0.303(29)\,\text{eV}, \quad \Gamma_{\text{PDG}} = 0.334(28)\,\text{eV}$$

$$g_{eff} = 0.425(13)\,\text{GeV}^{-1}$$



χPT NLO: L. Ametller, J. Bijnens, A. Bramon and F. Cornet, Phys. Lett. B **276** (1992) 185

Partial wave amplitudes

Coupled-channel once-subtracted dispersion relation for $I=1$ s-wave scattering:

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S-wave amplitudes:

$$k^{1,\text{Born}}_{0,++}(s) = e^2\frac{4m_K^2}{s\,\beta_K(s)}\log\frac{1+\beta_K(s)}{1-\beta_K(s)}, \quad h^{1,\text{V}}_{0,++}(s) = \sum_V 2e^2C_{12}\left(-\frac{M_V}{\beta_{\pi\eta}(s)}L_V(s)+s\right)$$

$$\beta_{ij}(s) = \frac{2p_{cm}}{\sqrt{s}}, \quad L_V(s) = \log\frac{X_V(s)+1}{X_V(s)-1} \quad \Longrightarrow \quad s_L = -\frac{(M_\rho^2-m_\pi^2)(M_\rho^2-m_\eta^2)}{M_\rho^2}$$

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D-wave amplitude:

$$h^2_{1,+-}(s) = -\frac{e^2\,C_{a_2\to\pi\eta}\,C_{a_2\to\gamma\gamma}\,s^2\,\beta^2_{\pi\eta}(s)}{10\,\sqrt{6}\,(s-M^2_{a_2}+i\,M_{a_2}\,\Gamma_{a_2}(s))}$$

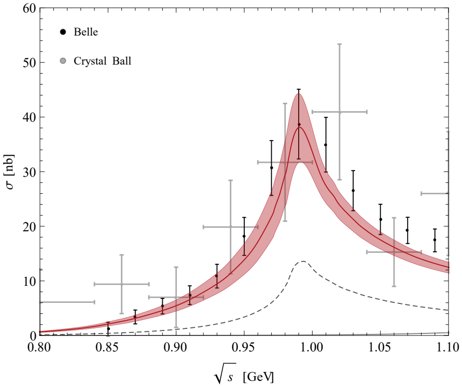
$$\Gamma_{a_2\to\pi\eta} = \frac{\beta^5_{\pi\eta}(M^2_{a_2})}{1920\,\pi}\,C^2_{a_2\to\pi\eta}\,M^3_{a_2} = 15.5(1.5)\,\mathrm{MeV}$$

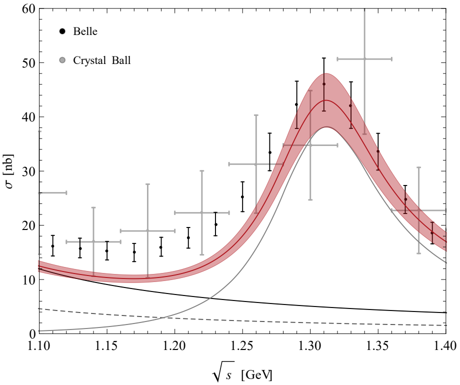
$$\Gamma_{a_2\to\gamma\gamma} = \frac{\pi\alpha^2}{5}\,C^2_{a_2\to\gamma\gamma}\,M^3_{a_2} = 1.0(1)\,\mathrm{keV}$$

D-wave amplitude: D. Drechsel, M. Gorchtein, B. Pasquini and M. Vanderhaeghen, Phys. Rev. C **61** (1999) 015204

Parameter free postdiction for $\gamma\gamma \to \pi^0\eta$

Differential cross section: $\frac{d\sigma}{d\cos\theta} = \frac{\beta_{\pi\eta}(s)}{64\,\pi\, s}\left(|H_{++}|^2 + |H_{+-}|^2\right)$



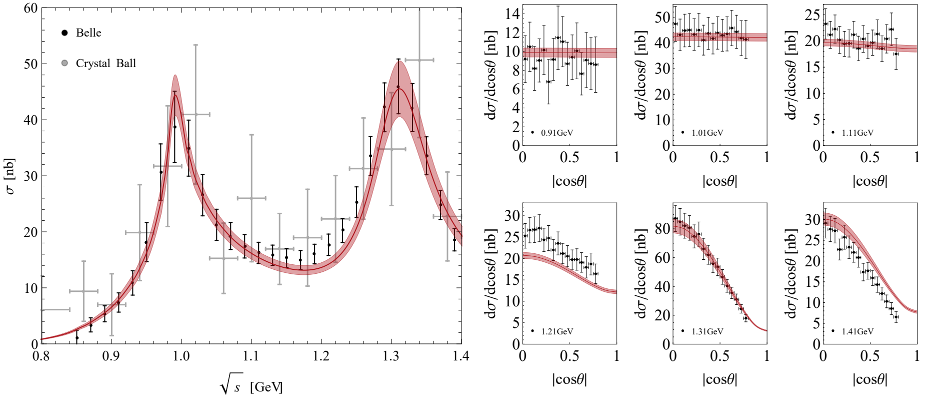


$|\cos\theta| < 0.8$ [Belle]

I. Danilkin, O. Deineka and M. Vanderhaeghen, Phys. Rev. D **96** (2017) no.11, 114018

Improved results for $\gamma\gamma \to \pi^0\eta$

Differential cross section: $\frac{d\sigma}{d\cos\theta} = \frac{\beta_{\pi\eta}(s)}{64\,\pi\,s}\left(|H_{++}|^2 + |H_{+-}|^2\right)$



$g_{eff} = 0.425(13)\ \text{GeV}^{-1}, \quad \Lambda_S = 1.46(6)\ \text{GeV}, \quad |\cos\theta| < 0.8$ [Belle]

I. Danilkin, O. Deineka and M. Vanderhaeghen, Phys. Rev. D **96** (2017) no.11, 114018

Pole position and two-photon width of $a_0(980)$

2 cuts $\Longrightarrow$ 4 Riemann sheets:

Sheet	Im $p_{\pi\eta}$	Im $p_{K\bar{K}}$
I	+	+
II	-	+
III	-	-
IV	+	-

Analytical continuation using unitarity, $\rho_i(s) = 2p_i(s)/\sqrt{s}$:

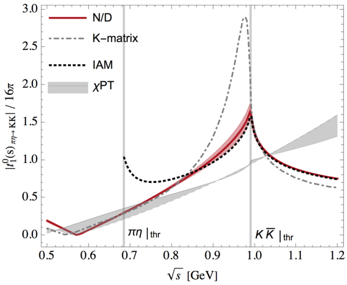
$$t^I(s+i\epsilon) - t^I(s-i\epsilon) = 2i\rho(s)t^I(s+i\epsilon)t^I(s-i\epsilon)$$

$$t^{II}(s-i\epsilon) \overset{\epsilon\to 0}{=} t^I(s+i\epsilon) = \frac{t^I(s-i\epsilon)}{1-2i\rho(s-i\epsilon)t^I(s-i\epsilon)}$$

Found a pole on IV Riemann sheet

$$\sqrt{s_{a_0}^{IV}} = (1.12^{-0.07}_{+0.02}) - \frac{i}{2}\,(0.28^{+0.08}_{-0.13})\ \text{GeV}$$

The residue leads to the couplings ratio $\left|c_{K\bar{K}}/c_{\pi\eta}\right| = 0.98^{-0.07}_{+0.20}$



N/D: I. V. Danilkin, L. I. R. Gil, and M. F. M. Lutz, Phys. Lett. **B703**, 504 (2011).
K-matrix: M. Albaladejo and B. Moussallam, Eur. Phys. J. **C77**,508 (2017).
IAM: A. Gomez Nicola and J. R. Pelaez, Phys. Rev. **D65**, 054009 (2002).
χPT: J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

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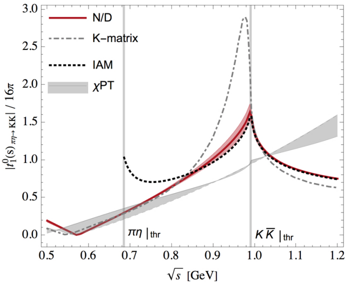

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χPT: J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

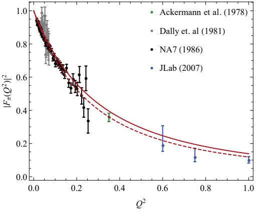
Two photon decay width: $\Gamma_{a_0\to\gamma\gamma} = \frac{|c_{\gamma\gamma}|^2}{16\pi M_{a_0}} = 0.27(4)\,\text{keV}$, PDG: $\Gamma_{a_0\to\gamma\gamma}\,\mathcal{B}(\pi^0\eta) = 0.21^{+0.08}_{-0.04}$ keV

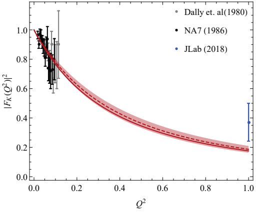
Single-virtual case: form factors

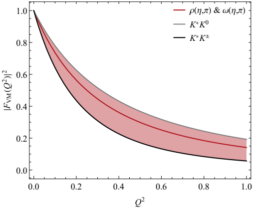
Consider now the process $\gamma\gamma^* \to \pi^0\eta$, where second photon has a **spacelike virtuality** $q_2^2 = -Q_2^2$

Form factors (VMD):

$$F_\pi(Q^2) = \frac{1}{1+Q^2/m_\rho^2} \overset{\text{data}}{\Longrightarrow} \frac{1}{1+Q^2/\Lambda_\pi^2}\,,\ \Lambda_\pi^2 = 0.525\pm0.008, \quad F_K(Q^2) = \frac{1/2}{1+Q^2/m_\rho^2} + \frac{1/6}{1+Q^2/m_\omega^2} + \frac{1/3}{1+Q^2/m_\phi^2} \overset{\text{data}}{\Longrightarrow} \frac{1}{1+Q^2/\Lambda_K^2}\,,\ \Lambda_K^2 = 0.760\pm0.081$$







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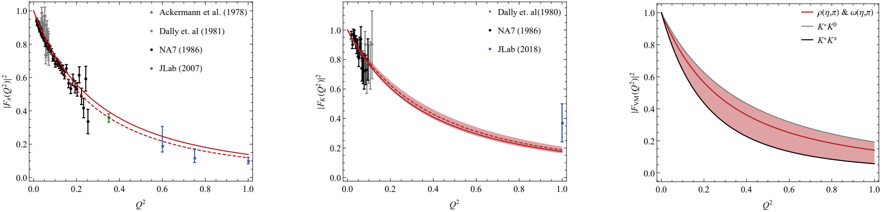
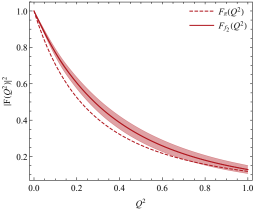


Form factor $a_2(1320)$:

$$F_{a_2}(Q^2) \approx F_{f_2}(Q^2) = \frac{1}{\left(1+Q^2/\Lambda_{f_2}^2\right)^2}\,, \Lambda_{f_2} = 1.222\pm0.066$$

or from sum rules:

$$F_{a_2}(Q^2) \approx F_\pi(Q^2)$$



$f_2(1270)$ form factor: M. Masuda *et al.* [Belle Collaboration], Phys. Rev. D **93** (2016) no.3, 032003
Light-by-light sum rules: V. Pascalutsa, V. Pauk and M. Vanderhaeghen, Phys. Rev. D **85** (2012) 116001

Single-virtual case: amplitudes

Invariant amplitudes: need additional Lorentz tensor $L_3^{\mu\nu} = -Q_2^2(q_1\cdot\Delta)g^{\mu\nu} + (q_1\cdot q_2)(q_2^\nu\Delta^\mu - q_1^\mu\Delta^\nu) - q_2^\mu q_2^\nu(q_1\cdot\Delta) + q_1^\mu q_1^\nu(q_2\cdot\Delta)$

$$F_1(s,t) = -\sum_V e^2\frac{C_{12}}{2}\left(\frac{4t+Q_2^2}{t-M_V^2} + \frac{4u+Q_2^2}{u-M_V^2}\right), \quad F_2(s,t) = \sum_V e^2\frac{C_{12}}{2}\left(\frac{1}{t-M_V^2} + \frac{1}{u-M_V^2}\right), \quad F_3(s,t) = -\sum_V e^2 C_{12}\left(\frac{1}{t-M_V^2} + \frac{1}{u-M_V^2}\right)$$

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Partial wave amplitudes:

S-wave: $k_{0,++}^{1,\text{Born}}(s) = \frac{e^2}{Q_2^2+s}\left(\frac{4m^2}{\beta_K(s)}\log\frac{1+\beta_K(s)}{1-\beta_K(s)}+2Q_2^2\right)$

$$h_{0,++}^{1,V}(s) = \sum_V e^2 C_{12}\Bigg[F_{V\pi}(Q_2^2)\Bigg(-\frac{L_{Vt}(s)}{\beta_{\pi\eta}(s)}\left(M_V^2+Q_2^2\left(\frac{M_V^2-m_\eta^2}{s+Q_2^2}\right)^2\right)+\frac{Q_2^2(M_V^2-m_\eta^2)}{s+Q_2^2} + \frac{Q_2^2(s+m_\pi^2-m_\eta^2)}{2s}+s\Bigg)+F_{V\eta}(Q_2^2)(m_\eta\leftrightarrow m_\pi)\Bigg]$$

P-wave: $$h_{1,++}^{1,V}(s) = \sum_V e^2 C_{12}\Bigg[F_{V\pi}(Q_2^2)\Bigg(-\frac{L_{Vt}(s)}{s\beta_{\pi\eta}^2(s)}\left(M_V^2+Q_2^2\left(\frac{M_V^2-m_\eta^2}{s+Q_2^2}\right)^2\right)\Bigg(s+m_\eta^2-m_\pi^2 + 2M_V^2 - 2\frac{sm_\eta^2+M_V^2Q_2^2}{s+Q_2^2}\Bigg)+\frac{2Q_2^2}{\beta_{\pi\eta}(s)}\left(\frac{M_V^2-m_\eta^2}{s+Q_2^2}\right)^2 + \frac{Q_2^2\beta_{\pi\eta}^2(s)+12M_V^2}{6\beta_{\pi\eta}(s)}\Bigg)-F_{V\eta}(Q_2^2)(m_\eta\leftrightarrow m_\pi)\Bigg]$$

D-wave: $h_{2,+-}(s) = -\frac{e^2}{10\sqrt{6}}C_{a_2\to\gamma\gamma}C_{a_2\to\pi\eta}\frac{s(s+Q_2^2)\beta_{\pi\eta}(s)^2}{s-m_{a_2}^2+im_{a_2}\Gamma_{a_2}}F_{a_2}(Q_2^2)$

Single-virtual case: amplitudes

Invariant amplitudes: need additional Lorentz tensor $L_3^{\mu\nu} = -Q_2^2(q_1\cdot\Delta)g^{\mu\nu} + (q_1\cdot q_2)(q_2^\nu\Delta^\mu - q_1^\mu\Delta^\nu) - q_2^\mu q_2^\nu(q_1\cdot\Delta) + q_1^\mu q_1^\nu(q_2\cdot\Delta)$

$$F_1(s,t) = -\sum_V e^2\frac{C_{12}}{2}\left(\frac{4t+Q_2^2}{t-M_V^2} + \frac{4u+Q_2^2}{u-M_V^2}\right),\quad F_2(s,t) = \sum_V e^2\frac{C_{12}}{2}\left(\frac{1}{t-M_V^2}+\frac{1}{u-M_V^2}\right),\quad F_3(s,t) = -\sum_V e^2 C_{12}\left(\frac{1}{t-M_V^2}+\frac{1}{u-M_V^2}\right)$$

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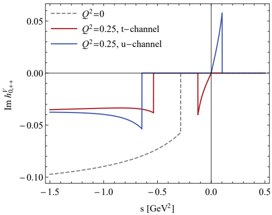
$$L_{Vt}(s) = \log\left(\frac{X_{Vt}(s)+1}{X_{Vt}(s)-1}\right)$$

$$X_V(s) = \frac{2M_V^2 - (m_\eta+m_\pi) + s + Q_2^2 \pm \frac{(m_\eta^2-m_\pi^2)Q_2^2}{s}}{(s+Q_2^2)\beta_{\pi\eta}(s)}$$

$$h_{0,++}^{1,\mathrm{V}}(s) = \sum_V e^2 C_{12}\Bigg[F_{V\pi}(Q_2^2)\Bigg(-\frac{L_{Vt}(s)}{\beta_{\pi\eta}(s)}\left(M_V^2 + Q_2^2\left(\frac{M_V^2-m_\eta^2}{s+Q_2^2}\right)^2\right) + \frac{Q_2^2(M_V^2-m_\eta^2)}{s+Q_2^2} + \frac{Q_2^2(s+m_\pi^2-m_\eta^2)}{2s} + s\Bigg) + F_{V\eta}(Q_2^2)(m_\eta\leftrightarrow m_\pi)\Bigg]$$



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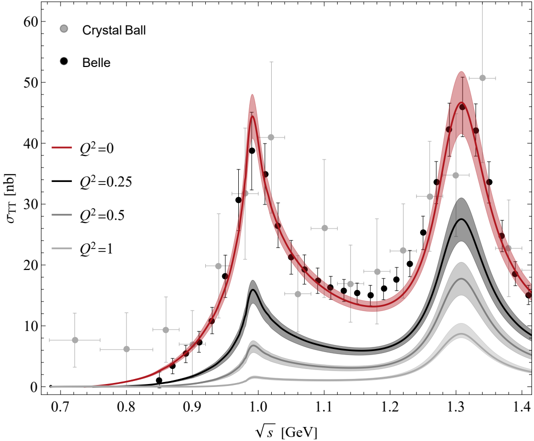
Preliminary results for $\gamma\gamma^* \to \pi^0\eta$

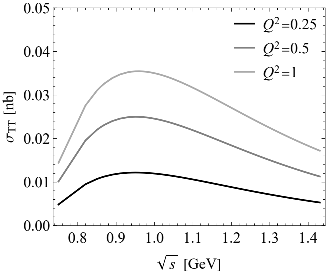


s- and d-waves

$$\sigma_{tot} = \sigma_{TT} + \epsilon_0 \sigma_{TL}$$

$$\frac{d\sigma_{TT}}{d\cos\theta} = \frac{\beta_{\pi\eta}(s)}{64\,\pi\, s\sqrt{X}}\,\frac{1}{2}\left(|H_{++}|^2 + |H_{+-}|^2\right)$$

$X \equiv (s + Q_2^2)^2/4$



p-wave

$g_{eff} = 0.425(13)$ GeV^{-1}, $\Lambda_S = 1.46(6)$ GeV $+$ $F_K(Q^2)$, $F_{\rm VMD}(Q^2)$, $F_{a_2}(Q^2) : [F_{f_2}(Q^2)$ or $F_\pi(Q^2)]$

Summary and Outlook

Summary

- **Dispersive** description of $\gamma\gamma \to \pi^0\eta$ process, including
  - Cross section and angular distribution consistent with the data from Belle Collaboration
  - Description of the cross channel $\eta \to \pi^0\gamma\gamma$ decay process with resulting $\Gamma(\eta \to \pi^0\gamma\gamma) = 0.303(29)\,\text{eV}$
  - **Pole position** of $a_0(980)$ on the IV Riemann sheet $\sqrt{s_{a_0}^{IV}} = \left(1.12^{-0.07}_{+0.02}\right) - \frac{i}{2}\left(0.28^{+0.08}_{-0.13}\right)$ GeV and **two photon decay width** $\Gamma_{a_0\to\gamma\gamma} = 0.27(4)\,\text{keV}$
- Dispersive **prediction** for $\gamma\gamma^* \to \pi\eta$ process for $Q^2 = 0.25, 0.5, 1$

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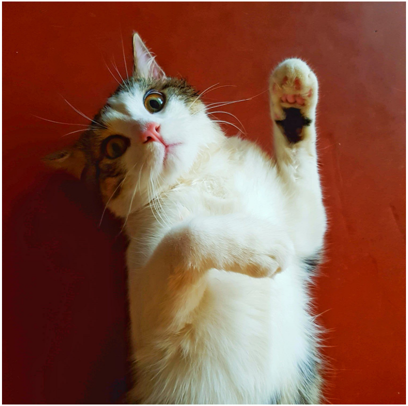
Outlook

- Analyze σ_{TL} contributions
- Compare results with the upcoming data from BESIII
- Provide an analysis of the double-virtual process $\gamma^*\gamma^* \to \pi^0\eta$ in s-wave
- Combine $I = 1$ result from $\gamma\gamma^{(*)} \to \pi^0\eta$ and $I = 0$ from $\gamma\gamma^{(*)} \to \pi\pi$ in order to analyze the $\gamma\gamma^{(*)} \to K\bar{K}$ process
- Calculate the contribution of the considered processes to $(g-2)_\mu$

$\gamma\gamma \to \pi^0\eta$: I. Danilkin, O. Deineka and M. Vanderhaeghen, Phys. Rev. D **96** (2017) no.11, 114018

$\gamma\gamma^* \to \pi\pi$: I. Danilkin and M. Vanderhaeghen, coming soon

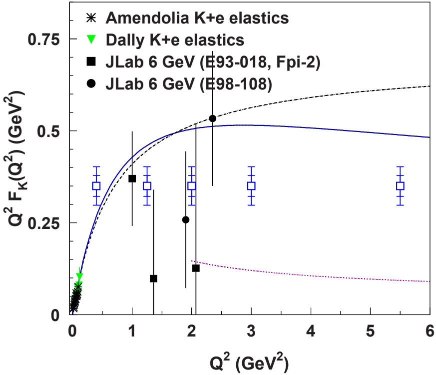
$\gamma^*\gamma^* \to \pi^0\eta$: O. Deineka, I. Danilkin and M. Vanderhaeghen, in preparation



The heaviest MESON ever found

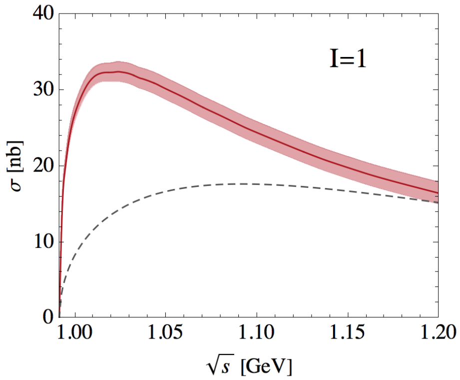
Thank you!

Additional: data for kaon TFF



M. Carmignotto *et al.*, Phys. Rev. C **97** (2018) no.2, 025204

Additional: $\gamma\gamma \to K\bar{K}$ isovector total cross section



Additional: vector TFF from VMD

$$F_{\omega\pi^0}(Q^2) = \frac{1/2}{1+Q^2/m_\rho^2},$$

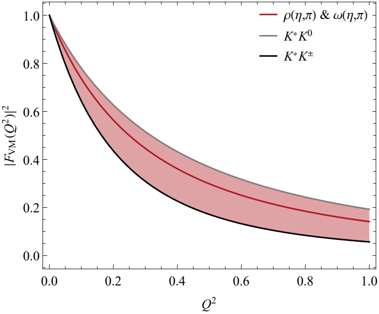
$$F_{\rho^0\pi^0}(Q^2) = \frac{1/2}{1+Q^2/m_\omega^2},$$

$$F_{\rho^0\eta}(Q^2) = \frac{1/2}{1+Q^2/m_\rho^2} + \frac{1/2}{1+Q^2/m_\omega^2},$$

$$F_{\omega\eta}(Q^2) = \frac{1/2}{1+Q^2/m_\rho^2} + \frac{1/2}{1+Q^2/m_\omega^2},$$

$$F_{K^{\pm *}K^{\pm}}(Q^2) = \frac{3/2}{1+Q^2/m_\rho^2} + \frac{1/2}{1+Q^2/m_\omega^2} - \frac{1}{1+Q^2/m_\phi^2},$$

$$F_{K^{0*}K^0}(Q^2) = \frac{3/4}{1+Q^2/m_\rho^2} - \frac{1/4}{1+Q^2/m_\omega^2} + \frac{1/2}{1+Q^2/m_\phi^2}$$



Additional: analytical expressions for the l.h.c.

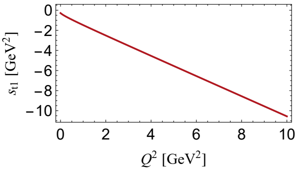
$$s_{t_1} = -\frac{1}{2M^2}\Big(M^4 + m_1^2(m_2^2 + Q_2^2) + M^2(-m_1^2 - m2^2 + Q_2^2)$$

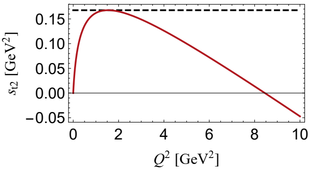
$$+ (M^2 - m_1^2)\sqrt{M^4 - 2M^2(m_2^2 - Q_2^2) + (m_2^2 + Q_2^2)^2}\Big) \xrightarrow{M\to\infty} -\infty\,,$$

$$s_{t_2} = -\frac{1}{2M^2}\Big(M^4 + m_1^2(m_2^2 + Q_2^2) + M^2(-m_1^2 - m2^2 + Q_2^2)$$

$$- (M^2 - m_1^2)\sqrt{M^4 - 2M^2(m_2^2 - Q_2^2) + (m_2^2 + Q_2^2)^2}\Big) \xrightarrow{M\to\infty} -0\,,$$

u-channel: $(m_1 \leftrightarrow m_2)$





Additional: hadronic input

N/D ansatz, numerically solved integral equations

$$N(s) = U(s) + \frac{s - s_{th}}{\pi} \int_{s_{th}}^{\infty} ds' \frac{N(s')\rho(s')(U(s') - U(s))}{(s' - s_{th})(s' - s)}$$

$$\Omega^{-1}(s) = 1 - \frac{s - s_{th}}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_{s_{th}}} \frac{N(s')\rho(s')}{s' - s}$$

Amplitude is reconstructed as

$$\boxed{T(s) = \Omega(s) N(s)}$$

Conformal mapping expansion

$$U(s) = \sum_k C_k \, \xi(s)^k, \quad C_k \text{ matched to } \chi\text{PT at threshold}$$

Λ_S dependence is explicitly given

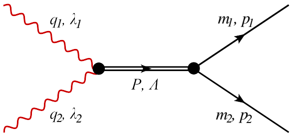
$$\xi(s) = \frac{a(\Lambda_S^2 - s)^2 - 1}{(a - 2b)(\Lambda_S^2 - s)^2 + 1}, \quad a = \frac{1}{(\Lambda_S^2 - \mu_E^2)^2}, \quad b = \frac{1}{(\Lambda_S^2 - \Lambda_0^2)^2}$$

I. V. Danilkin, L. I. R. Gil, and M. F. M. Lutz, Phys. Lett. **B703**, 504 (2011).

Additional: $a_2(1320)$ resonance

$a_2(1320)$ is taken into account as the explicit degrees of freedom:

$$\mathscr{L}_{T\to PP} = e^2 C_{T\to PP} T_{\mu\nu} F^{\mu\lambda} F^{\nu}_{\lambda}$$
$$\mathscr{L}_{T\to\gamma\gamma} = C_{T\to\gamma\gamma} T^{\mu\nu} \partial_\mu P \partial_\nu P$$



Helicity-2, d-wave:

$$h^2_{+-} = -\frac{e^2}{10\sqrt{6}} C_{T\to PP} C_{T\to\gamma\gamma} \frac{s^2 \beta_{\pi\eta}(s)^2}{s - m^2_{a_2} + i m_{a_2} \Gamma_{a_2}}$$

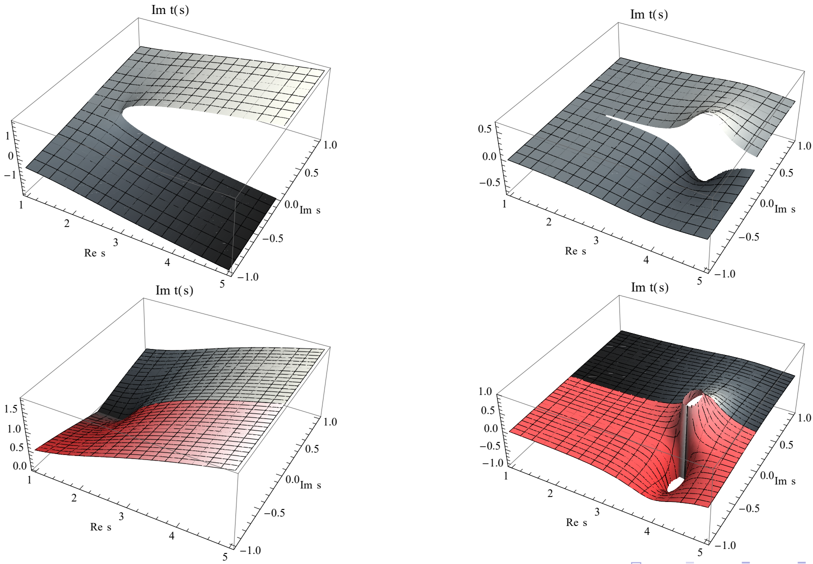
Couplings:

$$\Gamma_{a_2\to\pi\eta} = \frac{\beta^5_{\pi\eta}(M^2_{a_2})}{1920\,\pi} C^2_{a_2\to\pi\eta} M^3_{a_2} = 15.5(1.5)\,\text{MeV}$$
$$\Gamma_{a_2\to\gamma\gamma} = \frac{\pi\alpha^2}{5} C^2_{a_2\to\gamma\gamma} M^3_{a_2} = 1.0(1)\,\text{keV}$$

D-wave amplitude: D. Drechsel, M. Gorchtein, B. Pasquini and M. Vanderhaeghen, Phys. Rev. C **61** (1999) 015204

Additional: resonances on Riemann sheets

Resonance is the **pole of the amplitude** on unphysical Riemann sheet.



Additional: definition of the Riemann sheets

Number of Riemann sheets, 2^n cuts, consider the two channel process

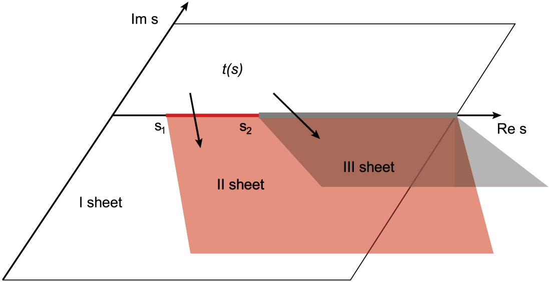
Coupled-channel unitarity

$$\text{Im}\, \boldsymbol{T}(s) = \boldsymbol{T}(s)\rho(s)\boldsymbol{T}^{\dagger}(s)$$

$$\boldsymbol{T}(s) = \begin{pmatrix} t_{11}(s) & t_{12}(s) \\ t_{12}(s) & t_{22}(s) \end{pmatrix}$$

$$\rho(s) = \begin{pmatrix} \rho_1(s) & 0 \\ 0 & \rho_2(s) \end{pmatrix}$$

Sheet	Im k_1	Im k_2
I	+	+
II	-	+
III	-	-
IV	+	-



plus IV sheet

Additional: analytical continuation using unitarity relation: II Riemann sheet

Unitarity relation in case of the two channels

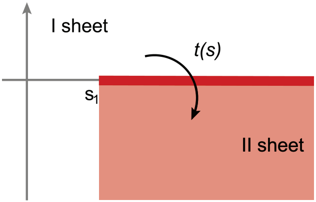
$$t^I_{11}(s+i\epsilon)-t^I_{11}(s-i\epsilon)=2i\rho_1(s)t^I_{11}(s+i\epsilon)t^I_{11}(s-i\epsilon)+2i\rho_2(s)t^I_{12}(s+i\epsilon)t^I_{12}(s-i\epsilon)$$
$$t^I_{12}(s+i\epsilon)-t^I_{12}(s-i\epsilon)=2i\rho_1(s)t^I_{11}(s+i\epsilon)t^I_{12}(s-i\epsilon)+2i\rho_2(s)t^I_{12}(s+i\epsilon)t^I_{22}(s-i\epsilon)$$
$$t^I_{22}(s+i\epsilon)-t^I_{22}(s-i\epsilon)=2i\rho_1(s)t^I_{12}(s+i\epsilon)t^I_{12}(s-i\epsilon)+2i\rho_2(s)t^I_{22}(s+i\epsilon)t^I_{22}(s-i\epsilon)$$

Extension to the **II Riemann sheet**

$$t^{II}_{11}(s-i\epsilon)\overset{\epsilon\to 0}{=}t^I_{11}(s+i\epsilon)=\frac{t^I_{11}(s-i\epsilon)}{1-2i\rho_1(s-i\epsilon)t^I_{11}(s-i\epsilon)}$$

$$t^{II}_{12}(s-i\epsilon)\overset{\epsilon\to 0}{=}t^I_{12}(s+i\epsilon)=\frac{t^I_{12}(s-i\epsilon)}{1-2i\rho_1(s-i\epsilon)t^I_{11}(s-i\epsilon)}$$

$$t^{II}_{22}(s-i\epsilon)\overset{\epsilon\to 0}{=}t^I_{22}(s-i\epsilon)+2i\rho_1(s)t^I_{12}(s+i\epsilon)t^I_{12}(s-i\epsilon)$$
$$=t^I_{22}(s-i\epsilon)+\frac{2i\rho_1(s-i\epsilon)t^I_{12}(s-i\epsilon)^2}{1-2i\rho_1(s-i\epsilon)t^I_{11}(s-i\epsilon)}$$



Additional: nalytical continuation using unitarity relation: III and IV Riemann sheets

III Riemann sheet

$$t^I_{11}(s+i\epsilon)-t^{II}_{11}(s-i\epsilon)=2i\rho_2(s)t^I_{12}(s+i\epsilon)t^{II}_{12}(s-i\epsilon)$$

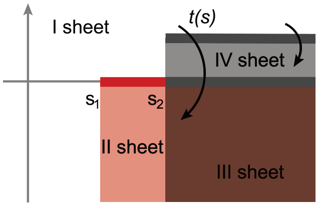
$$t^I_{12}(s+i\epsilon)-t^{II}_{12}(s-i\epsilon)=2i\rho_2(s)t^I_{12}(s+i\epsilon)t^{II}_{22}(s-i\epsilon)$$

$$t^I_{22}(s+i\epsilon)-t^{II}_{22}(s-i\epsilon)=2i\rho_2(s)t^I_{22}(s+i\epsilon)t^{II}_{22}(s-i\epsilon)$$

$$t^{III}_{11}(s-i\epsilon)=t^{II}_{11}(s-i\epsilon)+\frac{2i\rho_2(s)t^{II}_{12}(s-i\epsilon)^2}{1-2i\rho_2(s)t^{II}_{22}(s-i\epsilon)}$$

$$t^{III}_{12}(s-i\epsilon)=\frac{t^{II}_{12}(s-i\epsilon)^2}{1-2i\rho_2(s)t^{II}_{22}(s-i\epsilon)}$$

$$t^{III}_{22}(s-i\epsilon)=\frac{t^{II}_{22}(s-i\epsilon)^2}{1-2i\rho_2(s)t^{II}_{22}(s-i\epsilon)}$$



IV Riemann sheet

$$t^I_{11}(s+i\epsilon)-t^I_{11}(s-i\epsilon)=2i\rho_2(s)t^I_{12}(s+i\epsilon)t^I_{12}(s-i\epsilon)$$

$$t^I_{12}(s+i\epsilon)-t^I_{12}(s-i\epsilon)=2i\rho_2(s)t^I_{12}(s+i\epsilon)t^I_{22}(s-i\epsilon)$$

$$t^I_{22}(s+i\epsilon)-t^I_{22}(s-i\epsilon)=2i\rho_2(s)t^I_{22}(s+i\epsilon)t^I_{22}(s-i\epsilon)$$

$$t^{IV}_{11}(s-i\epsilon)=t^I_{11}(s-i\epsilon)+\frac{2i\rho_2(s)t^I_{12}(s-i\epsilon)^2}{1-2i\rho_2(s)t^I_{22}(s-i\epsilon)}$$

$$t^{IV}_{12}(s-i\epsilon)=\frac{t^I_{12}(s-i\epsilon)^2}{1-2i\rho_2(s)t^{II}_{22}(s-i\epsilon)}$$

$$t^{IV}_{22}(s-i\epsilon)=\frac{t^I_{22}(s-i\epsilon)^2}{1-2i\rho_2(s)t^{II}_{22}(s-i\epsilon)}$$

II sheet: $1-2i\rho_1(s)t^I_{11}(s)=0$

III sheet: $1-2i\rho_2(s)t^{II}_{22}(s)=0$

IV sheet: $1-2i\rho_2(s)t^I_{22}(s)=0$

Additional: two-photon couplings

For the $\gamma\gamma \to \pi\eta$ the pole is situated at IV Riemann sheet and the unitarity relation implies

$$t_{\gamma\gamma\to\pi\eta}(s+i\epsilon) - t_{\gamma\gamma\to\pi\eta}(s-i\epsilon) =$$
$$2i\rho_{\pi\eta}(s)t_{\gamma\gamma\to\pi\eta}(s-i\epsilon)t_{\pi\eta\to\pi\eta}(s+i\epsilon) + 2i\rho_{K\bar{K}}(s)t_{\gamma\gamma\to K\bar{K}}(s-i\epsilon)t_{K\bar{K}\to K\bar{K}}(s+i\epsilon)$$
$$t^{IV}_{\gamma\gamma\to\pi\eta}(s) - t^{I}_{\gamma\gamma\to\pi\eta}(s) = 2i\rho_{K\bar{K}}(s)t^{I}_{\gamma\gamma\to K\bar{K}}(s)t^{IV}_{K\bar{K}\to\pi\eta}(s)$$

In the vicinity of pole one can write

$$t^{IV}_{\gamma\gamma\to\pi\eta}(s) \simeq \frac{C_{\gamma\gamma}\,C_{\pi\eta}}{s^{\mathrm{IV}}_{a_0} - s}, \quad t^{IV}_{K\bar{K}\to\pi\eta}(s) \simeq \frac{C_{K\bar{K}}C_{\pi\eta}}{s^{\mathrm{IV}}_{a_0} - s}$$

The relation between couplings is then

$$\left(\frac{C_{\gamma\gamma}}{C_{K\bar{K}}}\right)^2 = -(2\rho_{K\bar{K}}(s_0))^2(t^{I}_{\gamma\gamma\to K\bar{K}}(s_0))^2$$

The two photon decay width

$$\Gamma_{\gamma\gamma} = \frac{|C_{\gamma\gamma}|^2}{16\pi m_0} = 0.27(4)\,\mathrm{keV}$$