A revision of radiative corrections to double-Dalitz decays ($P \to \bar{\ell}\ell\bar{\ell}'\ell'$)

based on K. Kampf, J. Novotny, PS: Phys.Rev. D97 (2018)

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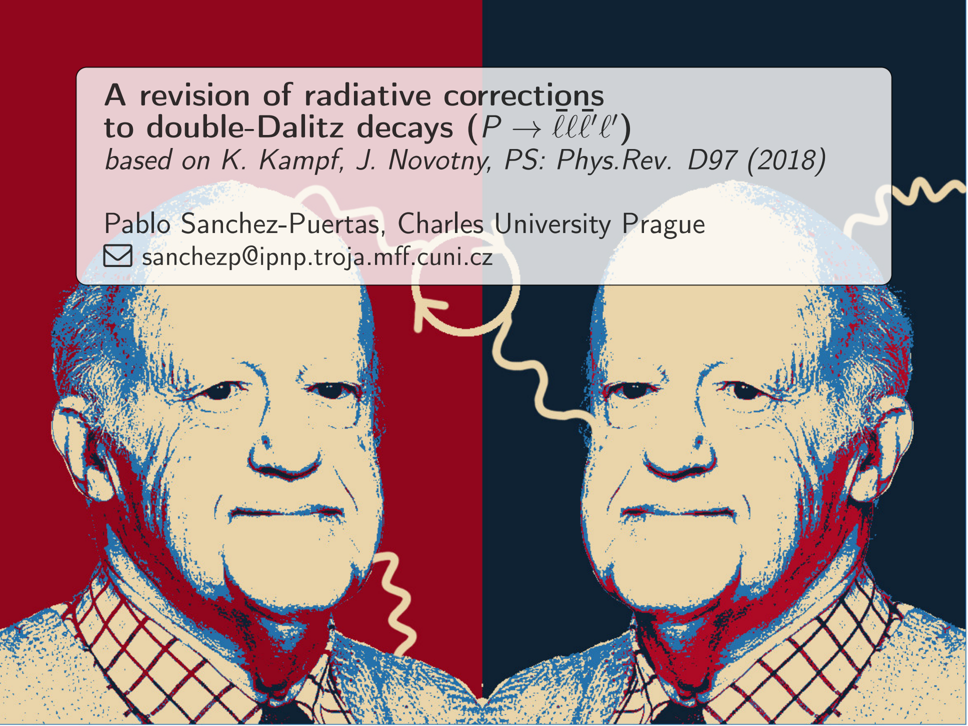

Outline

1. Motivation: the muon $g-2$ and HLbL
2. NLO QED corrections in $P \to \bar{\ell}\ell\bar{\ell}'\ell'$
3. Outlook: crossing to $e^+e^- \to e^+e^-P$

Section 1

Motivation: the muon $g - 2$ and HLbL

$(g-2)_\mu$: status and future

- $(g-2)_\mu$ anomaly might point to the existence of new physics (now)

$$\left.\begin{array}{l} a_\mu^{\mathrm{Exp.}} = 116592091(63)\times 10^{-11} \\ a_\mu^{\mathrm{Th}} = 116591776(44)\times 10^{-11} \end{array}\right] \Delta a_\mu = 315(77)\times 10^{-11} \longrightarrow \mathbf{4\sigma!}$$

$(g-2)_\mu$: status and future

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$$\left.\begin{array}{l} a_\mu^{\mathrm{FLab}} = 116592091(16) \times 10^{-11} \\ a_\mu^{\mathrm{Th}} = 116591783(44) \times 10^{-11} \end{array}\right] \Delta a_\mu = 306(76) \times 10^{-11} \longrightarrow \mathbf{4\sigma!}$$

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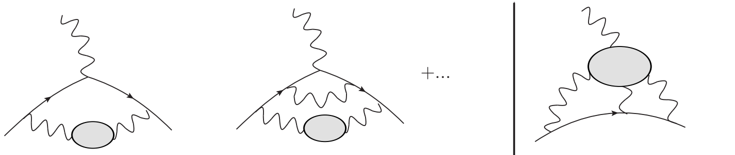
- We must improve theoretical errors: QCD-driven



$a_\mu^{\rm HVP,LO} = 6895(33) \times 10^{-11}$ $a_\mu^{\rm HVP,HO} = -87(1) \times 10^{-11}$ $a_\mu^{\rm HLbL} = 103(29) \times 10^{-11}$

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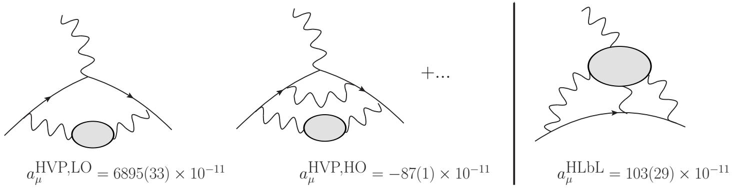
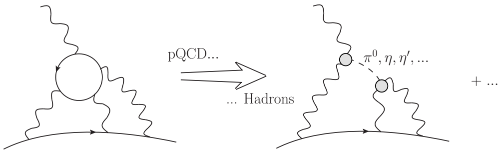


- As we shall see, this process relevant for the HLbL

$(g-2)_\mu$ & a theoretician's nightmare: HLbL

- QCD at m_μ scale non perturbative

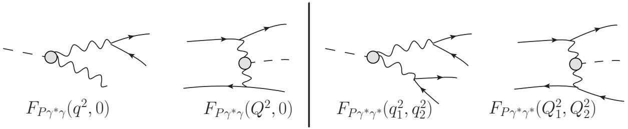


Illustrative Numbers	
$\boldsymbol{\pi^0, \eta, \eta'}$	**95(12)**
π, K loops	$-20(5)$
Scalar	$-6(1)$
Axials	8(3)
Tensor	1(0)
Quark loop	22(4)

- Need precise description for $\gamma^*\gamma^* M$ interactions: Form Factors

$$\text{eg. } i\mathcal{M}^{\mu\nu}_{\mathrm{PS}} = ie^2 \epsilon_{\mu\rho\nu\sigma} q_1^\rho q_3^\sigma F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$$

- Do it through data-based description



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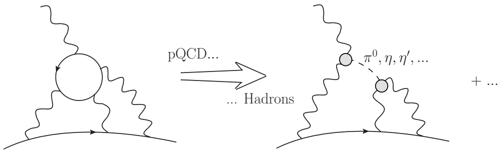

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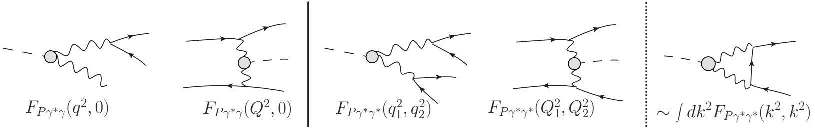

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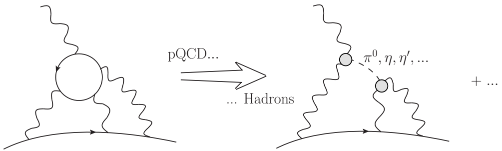

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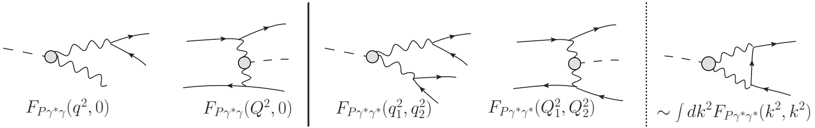


Precision requires appropiate care of (QED) RC!

Section 2

NLO QED corrections in $P \to \bar{\ell}\ell\bar{\ell}'\ell'$

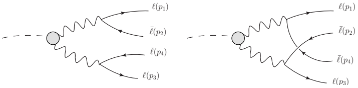
$P \to \bar{\ell}\ell\bar{\ell}'\ell'$: Towards NLO

Previous computation: Barker *et al*, PRD67, 2003

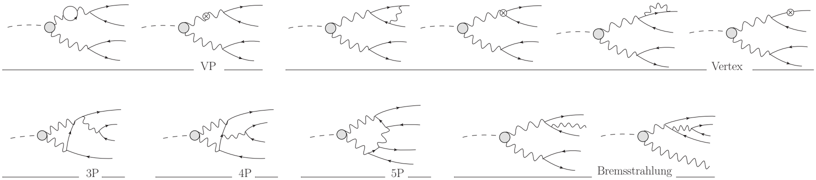
- Some suspicious points and missing diagrams
- Our goal: recompute (with soft-photon apprx.) and include 3P and 4P

Diagrams @NLO

- LO Contributions

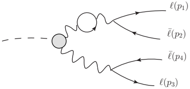


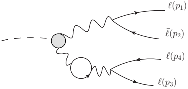
- NLO Contributions (+Exchange)



QED-universal

- Vacuum Polarization:





$$\mathcal{M}_D^{NLO} = \mathcal{M}_D^{LO}\Big(\Pi(s_{12}) + \Pi(s_{34})\Big) \quad ✓$$

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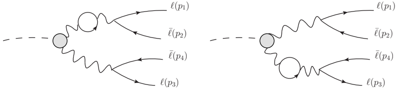

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$\Pi(s_{ij})$ analytic form: ✓

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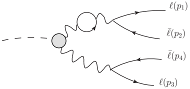

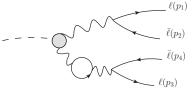

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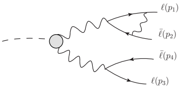
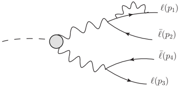


+



$$\bar{u}_i \Gamma^\mu v_j : \gamma^\mu \to \gamma^\mu F_1^{NLO} + i\frac{\sigma^{\mu\nu} q_\nu}{2m_\ell} F_2^{NLO}$$

F_1 PART: $\mathcal{M}_D^{NLO} = \mathcal{M}_D^{LO}[F_1(s_{12}) + F_1(s_{34})]$ ✓

F_2 PART: $\mathcal{M}_D^{NLO} \neq f(p_i, ..., p_j)\mathcal{M}_D^{LO} F_2$ (particularly $\neq \frac{2\mathcal{M}_D^{NLO}}{2-\beta_{ij}^2 \sin^2\theta_{ij}} F_2(s_{ij})$) ✗

Still, small differences

QED-universal

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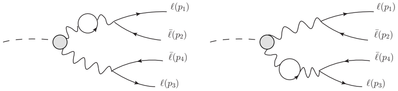

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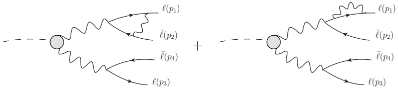
$\Pi(s_{ij})$ analytic form: ✓

- Vertex: F_1 ✓



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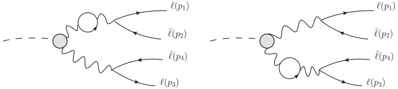

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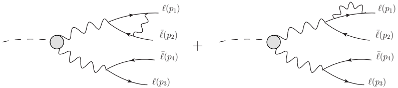

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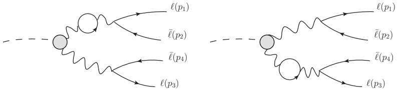

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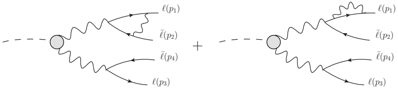

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Still, small differences

- Still a word of caution!

If exchange terms absent, $\delta \supset \sum_{s_{ij}} 2\,\mathrm{Re}\,\Pi(s_{ij}) + 2\,\mathrm{Re}\,F_1(s_{ij})$

$\Rightarrow$ not if exchange! (problems in original study?)

Bremsstrahlung: soft photon

- General result from 't Hooft & Veltman NPB153 (1979)

$$(i \neq j) \Rightarrow \frac{|\mathcal{M}^{\mathrm{LO}}|^2}{-\mathcal{Q}_i\mathcal{Q}_j}\frac{\alpha}{\pi}\frac{z_{ij}}{\lambda_{i,j}}\left[\ln\left(\frac{z_{i,j}+\lambda_{i,j}}{z_{i,j}-\lambda_{i,j}}\right)\ln\left(\frac{2E_c}{m_\gamma}\right)\right.$$
$$\left.+\frac{1}{4}\ln^2\left(\frac{u^0-\boldsymbol{u}}{u^0+\boldsymbol{u}}\right)+\mathrm{Li}_2\left(1-\frac{u^0-\boldsymbol{u}}{v}\right)+\mathrm{Li}_2\left(1-\frac{u^0+\boldsymbol{u}}{v}\right)\Bigg|_{u=p_j}^{u=\alpha p_i}\right].$$

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$$(p_i = p_j) \Rightarrow -|\mathcal{M}^{\mathrm{LO}}|^2\frac{\alpha}{\pi}\left[\ln\left(\frac{4E_c^2}{m_\gamma^2}\right)+\frac{p_i^0}{\boldsymbol{p_i}}\ln\left(\frac{p_i^0-\boldsymbol{p_i}}{p_i^0+\boldsymbol{p_i}}\right)\right]$$

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$$vs. \qquad -|\mathcal{M}^{\mathrm{LO}}|^2\frac{\alpha}{\pi}\left[\ln\left(\frac{2E_c}{m_\gamma}\right)+\frac{1}{2\lambda_{i,i}}\ln\left(\frac{1-\lambda_{i,i}}{1+\lambda_{i,i}}\right)\right] \qquad (\lambda_{i,i}\to 0)\ ✗\ ?$$

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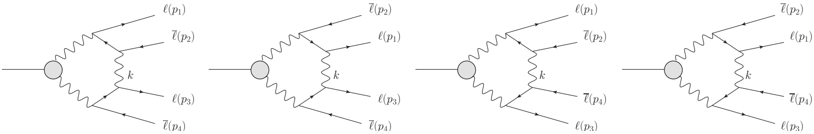
Still, taking last choice, less in agreement

QED "specific" 5P



$$i\mathcal{M}_{1D}^{5P} = -e^6 \int \frac{d^4k}{(2\pi)^4} \Big(-4(p_2\cdot p_3)(\bar{u}_1\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) + 2k_\alpha \Big[(\bar{u}_1\gamma^\nu\gamma^\alpha \not{p}_3 v_2)(\bar{u}_3\gamma^\sigma v_4) - (\bar{u}_1\gamma^\nu v_2)(\bar{u}_3 \not{p}_2 \gamma^\alpha\gamma^\sigma v_4) \Big] + k_\alpha k_\beta (\bar{u}_1\gamma^\nu\gamma^\alpha\gamma^\eta v_2)(\bar{u}_3\gamma_\eta\gamma^\beta\gamma^\sigma v_4)\Big) \mathcal{C}_{\nu\sigma}^{5P}(p_3,p_2), \checkmark$$

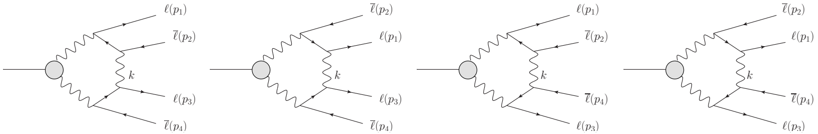
$$i\mathcal{M}_{2D}^{5P} = +e^6 \int \frac{d^4k}{(2\pi)^4} \Big(-4(p_1\cdot p_3)(\bar{u}_1\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) + 2k_\alpha \Big[(\bar{u}_1 \not{p}_3\gamma^\alpha\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) - (\bar{u}_1\gamma^\nu v_2)(\bar{u}_3 \not{p}_1 \gamma^\alpha\gamma^\sigma v_4) \Big] + k_\alpha k_\beta (\bar{u}_1\gamma^\eta\gamma^\alpha\gamma^\nu v_2)(\bar{u}_3\gamma_\eta\gamma^\beta\gamma^\sigma v_4)\Big) \mathcal{C}_{\nu\sigma}^{5P}(p_3,p_1) \checkmark$$

$$i\mathcal{M}_{3D}^{5P} = +e^6 \int \frac{d^4k}{(2\pi)^4} \Big(-4(p_2\cdot p_4)(\bar{u}_1\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) + 2k_\alpha \Big[(\bar{u}_1\gamma^\nu\gamma^\alpha \not{p}_4 v_2)(\bar{u}_3\gamma^\sigma v_4) - (\bar{u}_1\gamma^\nu v_2)(\bar{u}_3 \gamma^\sigma\gamma^\alpha \not{p}_2 v_4) \Big] + k_\alpha k_\beta (\bar{u}_1\gamma^\nu\gamma^\alpha\gamma^\eta v_2)(\bar{u}_3\gamma^\sigma\gamma^\beta\gamma_\eta v_4)\Big) \mathcal{C}_{\nu\sigma}^{5P}(p_4,p_2) \checkmark$$

$$i\mathcal{M}_{4D}^{5P} = -e^6 \int \frac{d^4k}{(2\pi)^4} \Big(-4(p_1\cdot p_4)(\bar{u}_1\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) + 2k_\alpha \Big[(\bar{u}_1 \not{p}_4\gamma^\alpha\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma v_4) - (\bar{u}_1\gamma^\nu v_2)(\bar{u}_3 \gamma^\sigma\gamma^\alpha \not{p}_1 v_4) \Big] + k_\alpha k_\beta (\bar{u}_1\gamma^\eta\gamma^\alpha\gamma^\nu v_2)(\bar{u}_3\gamma^\sigma\gamma^\beta\gamma_\eta v_4)\Big) \mathcal{C}_{\nu\sigma}^{5P}(p_4,p_1), \checkmark$$

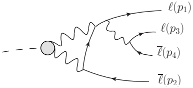
$$\mathcal{C}_{\nu\sigma}^{5P}(p_i,p_j) = \frac{\epsilon_{\mu\nu\rho\sigma}(p_{12}^\mu p_{34}^\rho + P^\mu k^\rho) F_{P\gamma\gamma}\left((k-p_{12})^2,(k+p_{34})^2\right)}{k^2[(k+p_i^2)-m_i^2](k+p_{34})^2(k-p_{12})^2[(k-p_j)^2-m_j^2]} \checkmark$$

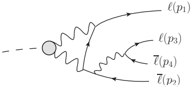
QED "specific" 5P

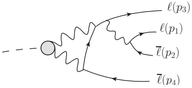


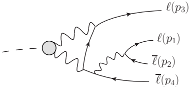
- No analytic expressions to check (would be complicated)
- Anyway, total correction vanishes unless identical leptons
- We used trace techniques vs. their spinor representation
- Allow to reduce down to 5P scalar and lower point tensor
- Also 5P tensor checked to give same result (different method wrt theirs)
- Integrals evaluated via LoopTools (5P-checked)
- Overall correction (MC integral) compatible with 0 as a check

QED "specific" 3P









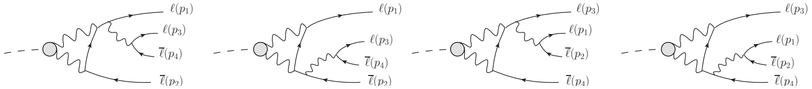
- Brand new result

$$i\mathcal{M}^{3P}_{1D} = \mathcal{C}_{3\text{P}} \int \frac{d^4k}{(2\pi)^4} \frac{[\bar{u}_1\gamma^\lambda(\not{p}_{134} + m_a)\Gamma_{3\text{P}}v_2][\bar{u}_3\gamma_\lambda v_4]}{k^2(k+P)^2((k+p_2)^2 - m_a^2)} \frac{F_{P\gamma\gamma}(k^2,(k+P)^2)}{p_{34}^2(p_{134}^2 - m_a^2)},$$

$$i\mathcal{M}^{3P}_{2D} = \mathcal{C}_{3\text{P}} \int \frac{d^4k}{(2\pi)^4} \frac{[\bar{u}_1\Gamma_{3\text{P}}(-\not{p}_{234} + m_a)\gamma^\lambda v_2][\bar{u}_3\gamma_\lambda v_4]}{k^2(k+P)^2((k+p_1)^2 - m_a^2)} \frac{F_{P\gamma\gamma}(k^2,(k+P)^2)}{p_{34}^2(p_{234}^2 - m_a^2)},$$

$$\mathcal{C}_{3\text{P}} = e^4\left(\frac{i}{16\pi^2}\right)^{-1}\frac{\alpha}{2\pi} \qquad \Gamma_{3\text{P}} = (k^2\not{P} - (k\cdot P)\not{k})\gamma^5$$

QED “specific” 3P



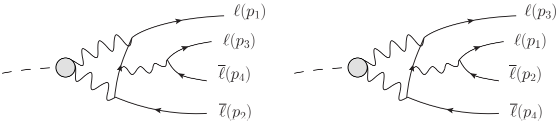
- Brand new result

$$2\,\mathrm{Re}\,\mathcal{M}_D^{LO*}\mathcal{M}_{1D}^{3P} = 2\,\mathrm{Re}\,\frac{e^8 F^*_{P\gamma\gamma}(s_{12},s_{34})}{x_{34}}\,\frac{\alpha}{4\pi}\Bigg[(\mathcal{I}_1-\mathcal{I}_2^a)\Bigg(4\lambda y_{12}(2+y_{34}^2-\lambda_{34}^2)-\frac{\lambda z y_{34}\Xi}{x_{12}x_{34}}$$
$$+\frac{2M^2\lambda^2(1-\lambda_{12}^2)(2+y_{34}^2-\lambda_{34}^2)}{p_{134}^2-m_a^2}\Bigg)-\mathcal{I}_2^b\left(\frac{M^2\lambda^2(1-\lambda_{12}^2)(2+y_{34}^2-\lambda_{34}^2)}{p_{134}^2-m_a^2}\right)\Bigg]$$

$$\mathcal{I}_1 = B_0(p_{134}^2, M_{V_2}^2, m_a^2) + M_{V_1}^2\, C_0(M^2, p_{134}^2, m_a^2, M_{V_1}^2, M_{V_2}^2, m_a^2)$$
$$\mathcal{I}_2^a = C_{00} + M^2 C_{11} + (p_2\cdot P)C_{12}, \quad \mathcal{I}_2^b = M^2 C_{12} + (p_2\cdot P)C_{22}$$

- UV-div for a constant TFF: include it or counterterm $P \to \bar{\ell}\ell$

QED "specific" 4P

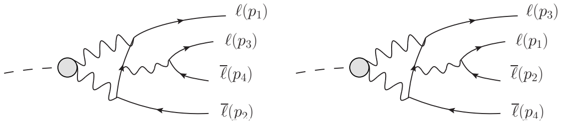


- Brand new result

$$i\mathcal{M}_{\rm 1D}^{\rm 4P} = \mathcal{C}_{\rm 4P}\int \frac{d^4k}{(2\pi)^4}\frac{\left[\bar{u}_1\Gamma^{\lambda}_{\rm 4P}v_2\right]\left[\bar{u}_3\gamma_{\lambda}v_4\right]F_{P\gamma\gamma}(k^2,(k+P)^2)}{k^2[(k+p_1)^2-m_a^2][(k+p_{134})^2-m_a^2](k+P)^2}\frac{1}{s_{34}},$$

$$\Gamma^{\lambda}_{\rm 4P} = 2i\left(k^{\lambda}(k+P)^2\not{k}-(k+P)^{\lambda}k^2(\not{k}+\not{P})\right)\gamma^5 + 2\epsilon_{\mu\nu\rho\sigma}k^{\mu}P^{\rho}\left(p_1^{\nu}\gamma^{\lambda}(\not{k}+\not{P})\gamma^{\sigma}+p_2^{\nu}\gamma^{\sigma}\not{k}\gamma^{\lambda}\right),$$

QED “specific” 4P

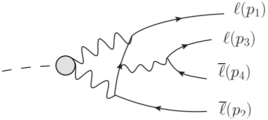


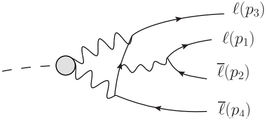
- Brand new result

$$-2\,\mathrm{Re}\,\mathcal{M}_D^{LO*}\mathcal{M}_{1D}^{4P} = \mp 2e^8\,\mathrm{Re}\,\frac{F^*_{P\gamma^*\gamma^*}(s_{12(14)}, s_{34(32)})}{s_{12}s_{34}^2(s_{14}s_{34}s_{32})}\frac{\alpha}{4\pi}\left([\ldots]_1 + [\ldots]_2 + [\ldots]_3\right),$$

$$\epsilon_{\mu\nu\rho\sigma}p_{12}^{\mu}p_{34}^{\rho} \times \mathrm{tr}(\not{p}_1 + m_a)\Gamma^{\lambda(i)}_{4\mathrm{P}}(\not{p}_2 - m_a)\gamma^{\nu} \times \mathrm{tr}(\not{p}_3 + m_b)\gamma_{\lambda}(\not{p}_4 - m_b)\gamma^{\sigma},$$

QED "specific" 4P

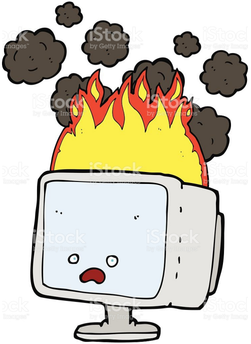




- Brand new result

$$
\begin{aligned}
\Gamma^{\lambda(1)}_{\rm 4P} &= 2i\Big[\gamma^\lambda C_{00} + m_a p_1^\lambda\left(C_{11}+2C_{12}+C_{22}\right) + p_1^\lambda \not{p}_{34}\left(C_{12}+C_{22}\right)\Big]\gamma^5 + 2iM^2_{V_2}\Big[\gamma^\lambda D_{00} \\
&\quad + m_a p_1^\lambda\left(D_{11}+2D_{12}+D_{22}+3D_{13}+3D_{23}+2D_{33}\right) + p_1^\lambda \not{p}_{34}\Big(D_{12}+D_{13}+D_{22} \\
&\quad + 2D_{23}+D_{33}\Big) + m_a p_2^\lambda\left(D_{13}+D_{23}+2D_{33}\right) + p_2^\lambda \not{p}_{34}\left(D_{23}+D_{33}\right)\Big]\gamma^5 - (p_1 \to p_2), \\
\Gamma^{\lambda(2)}_{\rm 4P} &= 2\epsilon_{\mu\nu\rho\sigma} P^\rho p_1^\nu \gamma^\lambda\gamma^\mu\gamma^\sigma D_{00} + 2\epsilon_{\mu\nu\rho\sigma} p_{34}^\mu p_1^\nu p_2^\rho \Big(\gamma^\lambda \not{p}_1 \gamma^\sigma\left(D_{12}+D_{22}+D_{23}+D_2\right) \\
&\quad + m_a \gamma^\lambda\gamma^\sigma\left(D_{23}+D_2\right) + \gamma^\lambda \not{p}_{34}\gamma^\sigma\left(D_{22}+D_{23}+D_2\right)\Big), \\
\Gamma^{\lambda(3)}_{\rm 4P} &= 2\epsilon_{\mu\nu\rho\sigma} P^\rho p_2^\nu \gamma^\sigma\gamma^\mu\gamma^\lambda D_{00} + 2\epsilon_{\mu\nu\rho\sigma} p_{34}^\mu p_1^\nu p_2^\rho \Big(\gamma^\sigma \not{p}_2 \gamma^\lambda D_{13} + \gamma^\sigma \not{p}_{34}\gamma^\lambda\left(D_{13}+D_{12}\right) \\
&\quad - m_a \gamma^\sigma\gamma^\lambda\left(D_{11}+D_{12}+D_{13}\right)\Big).
\end{aligned}
$$

And we are done ... so evaluate the full BR!
This is, MC integration



$P \to \bar{\ell}\ell\bar{\ell}'\ell'$ @NLO Summary

	$\pi^0 \to 4e$	$K_L \to 4e$	$K_L \to 2e2\mu$	$K_L \to 4\mu$	$\eta \to 4e$	$\eta \to 2e2\mu$	$\eta \to 4\mu$
δ(NLO)	−0.1727(2)	−0.2345(1)	−0.0842(2)	0.0608(2)	−0.2409(1)	−0.0900(1)	0.0455(2)
δ(FF)	0.0037(2)	0.0749(2)	0.6942(2)	0.8608(3)	0.0207(2)	0.4829(2)	0.6202(3)
~~3P,4P~~	−0.1718(2)	−0.2262(2)	−0.0767(1)	0.0704(1)	−0.2301(1)	−0.0836(1)	0.0535(1)
Barker	−0.160(2)	−0.218(1)	−0.066(1)	0.084(1)	–	–	–
BR(NLO)	$2.840(1)10^{-5}$	$5.120(1)10^{-5}$	$4.436(1)10^{-6}$	$1.851(1)10^{-9}$	$5.202(1)10^{-5}$	$5.393(1)10^{-6}$	$10.289(2)10^{-9}$

- Difference of 1% order wrt Barker *et al* for ~~3P,4P~~
- 3P,4P of 1% order roughly (except π^0)
- Could not certainly trace to the origin of differences

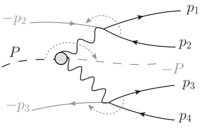
Status & Future

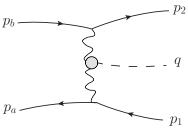
- For π^0 (30k Ev., KTeV) PRL100 (2008) negative slope (old RC related?)
- Our results (soft photon) coded for NA62 (T. Husek)
- For η (362 Ev., KLOE-2) PLB702 (2011) small statistics
- Proposed REDTOP exp: $10^{13(11)}$ $\eta^{(\prime)}$ interesting

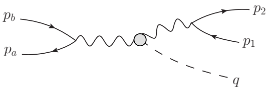
Section 3

Outlook: crossing to $e^+e^- \to e^+e^-P$

Outlook: crossing to $e^+e^- \to e^+e^- P$







- Double-virtual effects in double-Dalitz elusive for π^0 and some time for η
- Go to crossing-related $[e^+e^- \to e^+e^- P]$ @NLO (thanks to A. Kupsc)
- Dictionary for relating crossed-related variables relevant there ✓
- Recompute for t-channel (incl. bremsstrahlung) in a convenient form ✓
- Newly-released EKHARA 3.0 MC includes LO s, t-channel and NLO universal t-channel with soft and hard photon (H. Czyz)
- In contact with H. Czyz to check missing parts


Thanks
for
your
attention

			$\pi^0 \to e^+e^-e^+e^-$				
D+E	0.0392(2)	−0.0032(2)	−0.6391(6)	−0.0007(0)	−0.0017(1)	0	−0.6055(7)
Int	−0.0008(1)	0.0000(0)	0.0126(1)	−0.0004(0)	0.0005(0)	0.0009(1)	0.0128(2)
Total	0.0384(2)	−0.0032(2)	−0.6265(6)	−0.0011(0)	−0.0012(1)	0.0009(1)	−0.5927(7)
			$K_L^0 \to e^+e^-e^+e^-$				
D+E	0.1047(1)	−0.0045(0)	−1.6890(5)	−0.0016(1)	−0.0048(5)	0	−1.5952(7)
Int	−0.0016(1)	0.0000(0)	0.0265(3)	−0.0012(1)	0.0013(1)	0.0017(3)	0.0267(4)
Total	0.1031(1)	−0.0045(0)	−1.6625(6)	−0.0028(1)	−0.0035(5)	0.0017(3)	−1.5685(9)
			$K_L^0 \to e^+e^-\mu^+\mu^-$				
D	0.1067(1)	−0.0107(1)	−0.4763(5)	−0.0067(2)	−0.0209(2)	0	−0.4079(8)
			$K_L^0 \to \mu^+\mu^-\mu^+\mu^-$				
D+E	0.0481(0)	−0.0080(0)	0.0985(1)	−0.0027(1)	−0.0070(2)	0	0.1289(2)
Int	−0.0026(2)	0.0004(0)	−0.0044(0)	−0.0013(0)	−0.0007(0)	−0.0142(1)	−0.0228(2)
Total	0.0455(2)	−0.0076(0)	0.0941(1)	−0.0040(1)	−0.0077(2)	−0.0142(1)	0.1061(3)
			$\eta \to e^+e^-e^+e^-$				
D+E	0.1086(1)	−0.0045(0)	−1.7490(5)	−0.0016(1)	−0.0044(4)	0	−1.6509(6)
Int	−0.0015(1)	0.0000(0)	0.0251(2)	−0.0011(1)	0.0012(1)	0.0015(2)	0.0016(6)
Total	0.1070(1)	−0.0045(0)	−1.7239(5)	−0.0027(1)	−0.0032(4)	0.0015(2)	−1.6509(6)
			$\eta \to e^+e^-\mu^+\mu^-$				
D	0.1337(1)	−0.0127(1)	−0.6267(6)	−0.0057(1)	−0.0224(2)	0	−0.5338(7)
			$\eta \to \mu^+\mu^-\mu^+\mu^-$				
D+E	0.2914(3)	−0.0446(0)	0.3679(4)	−0.0111(3)	−0.0361(11)	0	0.5675(12)
Int	−0.0229(2)	0.0035(1)	−0.0207(2)	−0.0056(2)	−0.0018(1)	−0.0718(6)	−0.1193(7)
Total	0.2685(4)	−0.0411(1)	0.3472(4)	−0.0167(4)	−0.0379(11)	−0.0718(6)	0.4482(15)
	VP	F_2	F_1	3P	4P	5P	NLO